Mémoire d'Habilitation à Diriger des Recherches
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Quelques contributions à la topologie et à l'arithmétique des polynômes

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Invariance of Milnor numbers and topology of complex polynomials

INVARIANCE OF MILNOR NUMBERS AND TOPOLOGY OF COMPLEX POLYNOMIALS

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Abstract. We give a global version of Lê-Ramanujam μ-constant theorem for polynomials. Let (f_t), $t \in [0,1]$, be a family of polynomials of n complex variables with isolated singularities, whose coefficients are polynomials in t. We consider the case where some numerical invariants are constant (the affine Milnor number $\mu(t)$, the Milnor number at infinity $\lambda(t)$, the number of critical values, the number of affine critical values, the number of critical values at infinity). Let $n = 2$, we also suppose the degree of the f_t is a constant, then the polynomials f_0 and f_1 are topologically equivalent. For $n > 3$ we suppose that critical values at infinity depend continuously on t, then we prove that the geometric monodromy representations of the f_t are all equivalent.

1. Introduction

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geqslant 2$. By a result of Thom [Th] there is a finite minimal set of complex numbers $\mathcal{B}$, the *critical values*, such that $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a fibration.

1.1. **Affine singularities.** We suppose that *affine singularities are isolated i.e.* that the set $\{x \in \mathbb{C}^n \mid \operatorname{grad}_f x = 0\}$ is a finite set. Let μ_c be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathit{aff}} = \big\{c \mid \mu_c > 0\big\} \quad \text{and} \quad \mu = \sum_{c \in \mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*.

1.2. **Singularities at infinity.** See [Br]. Let d be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, let $f = f^d + f^{d-1} + \cdots + f^0$ where f^j is homogeneous of degree j. Let $\bar{f}(x, x_0)$ (with $x = (x_1, \ldots, x_n)$) be the homogenization of f with the new variable x_0: $\bar{f}(x, x_0) = f^d(x) + f^{d-1}(x)x_0 + \ldots + f^0(x)x_0^d$. Let

$$X = \Big\{((x : x_0), c) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x, x_0) - c x_0^d = 0\Big\}.$$

Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(x_0 = 0)$. The singular locus of X has the form $\Sigma \times \mathbb{C}$ where

$$\Sigma = \left\{ (x:0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0 \right\} \subset \mathcal{H}_\infty.$$

We suppose that f has *isolated singularities at infinity* that is to say that Σ is finite. This is always true for $n = 2$. For $n \geqslant 2$ such polynomials have been studied by S. Broughton [Br] and by A. Parusiński [Pa]. For a point $(x : 0) \in \mathcal{H}_\infty$, assume, for example, that $x = (x_1, \ldots, x_{n-1}, 1)$. Set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, x_0) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - c x_0^d.$$

Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of F_c at the point $(\check{x}, 0)$. If $(x : 0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic s, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many c, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Let

$$\mathcal{B}_\infty = \left\{ c \in \mathbb{C} \mid \lambda_c > 0 \right\} \quad \text{and} \quad \lambda = \sum_{c\in\mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity.* We can now describe the set of critical values $\mathcal{B}$ as follows (see [HL] and [Pa]):

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

Moreover, by [HL] and [ST], for $s \notin \mathcal{B}$, $f^{-1}(s)$ has the homotopy type of a wedge of $\lambda + \mu$ spheres of real dimension $n - 1$.

1.3. **Statement of the results.**

Theorem 1. *Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ whose coefficients are polynomials in t. We suppose that affine singularities and singularities at infinity are isolated. Let suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{\mathit{aff}}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t \in [0,1]$. Moreover let us suppose that critical values at infinity $\mathcal{B}_\infty(t)$ depend continuously on t. Then the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_1 : f_1^{-1}(\mathbb{C} \setminus \mathcal{B}(1)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(1)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic.*

Remark 1. As a consequence for $n \neq 3$ and $* \notin \mathcal{B}(0) \cup \mathcal{B}(1)$ the monodromy representations

$$\pi_1(\mathbb{C} \setminus \mathcal{B}(0), *) \longrightarrow \mathrm{Diff}(f_0^{-1}(*)) \text{ and}$$
$$\pi_1(\mathbb{C} \setminus \mathcal{B}(1), *) \longrightarrow \mathrm{Diff}(f_1^{-1}(*))$$

are equivalent (where $\mathrm{Diff}(f_t^{-1}(*))$ denotes the diffeomorphisms of $f_t^{-1}(*)$ modulo diffeomorphisms isotopic to identity).

Remark 2. The restriction $n \neq 3$, as in [LR], is due to the use of the h-cobordism theorem. The proof is based on the articles of Hà H.V.-Pham T.S. [HP] and of Lê D.T.-C.P. Ramanujam [LR].

Remark 3. This result extends a theorem of Hà H.V and Pham T.S. [HP] which deals only with monodromy at infinity (which corresponds to a loop around the whole set $\mathcal{B}(t)$) for $n = 2$. For $n \neq 3$, the fact that the monodromies at infinity are diffeomorphic is proved in [HZ] (for M-tame polynomials, with affine Milnor number constant) and in [Ti] (for generic fibers with homotopy type equivalent to a fixed number of $(n-1)$-spheres, with the hypothesis that $\mathcal{B}(t)$ is included in a compact set for all t).

Lemma 2. *Under the hypotheses of the previous theorem (except the hypothesis of continuity of the critical values), and one of the following conditions:*

- $n = 2$, *and* $\deg f_t$ *does not depend on* t*;*
- $\deg f_t$*, and* $\Sigma(t)$ *do not depend on* t*, and for all* $(x : 0) \in \Sigma(t)$*,* $\nu_{\tilde{x}}(t)$ *is independent of* t*;*

we have that $\mathcal{B}_\infty(t)$ *depends continuously on* t*,* i.e. *if* $c(\tau) \in \mathcal{B}_\infty(\tau)$ *then for all* t *near* τ *there exists* $c(t)$ *near* $c(\tau)$ *such that* $c(t) \in \mathcal{B}_\infty(t)$*.*

Under the hypothesis that there is no singularity at infinity we can prove the stronger result:

Theorem 3. *Let* $(f_t)_{t\in[0,1]}$ *be a family of complex polynomials whose coefficients are polynomials in* t*. Suppose that* $\mu(t)$*,* $\#\mathcal{B}_{\mathit{aff}}(t)$ *do not depend on* $t \in [0,1]$*. Moreover suppose that* $n \neq 3$ *and for all* $t \in [0,1]$ *we have* $\mathcal{B}_\infty(t) = \varnothing$*. Then the polynomials* f_0 *and* f_1 *are topologically equivalent, that is to say, there exist homeomorphisms* Φ *and* Ψ *such that*

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\Phi} & \mathbb{C}^n \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathbb{C} & \xrightarrow[\Psi]{} & \mathbb{C}. \end{array}$$

For the proof we glue the former study with the version of the μ-constant theorem of Lê D.T. and C.P. Ramanujam stated by J.G. Timourian [Tm]: a μ-constant deformation of germs of isolated hypersurface singularity is a product family.

For polynomials in two variables we can prove the following theorem which is a global version of Lê-Ramanujam-Timourian theorem:

Theorem 4. *Let* $n = 2$*. Let* $(f_t)_{t\in[0,1]}$ *be a family of complex polynomials whose coefficients are polynomials in* t*. Suppose that the integers*

$\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{\mathit{aff}}(t)$, $\#\mathcal{B}_\infty(t)$, $\deg f_t$ do not depend on $t \in [0,1]$. Then the polynomials f_0 and f_1 are topologically equivalent.

It uses a result of L. Fourrier [Fo] that gives a necessary and sufficient condition for polynomials to be topologically equivalent outside sufficiently large compact sets of $\mathbb{C}^2$.

Remark 4. In theorems 3 and 4 not only f_0 and f_1 are topologically equivalent but we can prove that it is a topologically trivial family.

This work was initiated by an advice of Lê D.T. concerning the article [Bo]: "It is easier to find conditions for polynomials to be equivalent than find all polynomials that respect a given condition."

We will denote $B_R = \big\{x \in \mathbb{C}^n \mid \|x\| \leqslant R\big\}$, $S_R = \partial B_R = \big\{x \in \mathbb{C}^n \mid \|x\| = R\big\}$ and $D_r(c) = \big\{s \in \mathbb{C} \mid |s-c| \leqslant r\big\}$.

2. Fibrations

In this paragraph we give some properties for a complex polynomial of n variables. The two first lemmas are consequences of transversality properties. There are direct generalizations of lemmas of [HP]. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial with isolated affine singularities and with isolated singularities at infinity. Let choose $r > 0$ such that $\mathcal{B}$ is contained in the interior of $D_r(0)$. For each fiber $f^{-1}(c)$ there is a finite number of real numbers $R > 0$ such that $f^{-1}(c)$ has non-transversal intersection with the sphere S_R (see [M3], Corollaries 2.8 and 2.9). So, for a sufficiently large number $R(c)$, the intersection $f^{-1}(c)$ with S_R is transversal for all $R \geqslant R(c)$. Let R_1 be greater than the maximum of the $R(c)$ with $c \in \mathcal{B}$, we also choose $R_1 \gg r$. We choose a small ε, $0 < \varepsilon \ll 1$ such that for all values c in the bifurcation set $\mathcal{B}$ of f and for all $s \in D_\varepsilon(c)$ the intersection $f^{-1}(s) \cap S_{R_1}$ is transversal, this is possible by continuity of the transversality. We denote

$$K = D_r(0) \setminus \bigcup_{c \in \mathcal{B}} \mathring{D}_\varepsilon(c).$$

Lemma 5. *There exists $R_0 \gg 1$ such that for all $R \geqslant R_0$ and for all s in K, $f^{-1}(s)$ intersects S_R transversally.*

Proof. We have to adapt the beginning of the proof of [HP]. If the assertion is false then we have a sequence (x_k) of points of $\mathbb{C}^n$ such that $f(x_k) \in K$ and $\|x_k\| \to +\infty$ as $k \to +\infty$ and such that there exist complex numbers λ_k with $\operatorname{grad}_f x_k = \lambda_k x_k$, where the gradient is Milnor gradient: $\operatorname{grad}_f = \left(\overline{\frac{\partial f}{\partial x_1}}, \ldots, \overline{\frac{\partial f}{\partial x_n}}\right)$. Since K is a compact set we can suppose (after extracting a sub-sequence, if necessary) that $f(x_k) \to c \in K$ as $k \to +\infty$. Then by the Curve Selection Lemma of [NZ] there exists a real analytic curve $x :]0, \varepsilon[\longrightarrow \mathbb{C}^n$ such that $x(\tau) = a\tau^\beta + a_1\tau^{\beta+1} + \cdots$ with $\beta < 0$, $a \in \mathbb{R}^{2n} \setminus \{0\}$ and $\operatorname{grad}_f x(\tau) = \lambda(\tau)x(\tau)$. Then $f(x(\tau)) = c + c_1\tau^\rho + \cdots$ with $\rho > 0$. So $f(x(\tau)) \to c$ as $\tau \to 0$. Then we can redo the calculus of [HP]:

$$\frac{df(x(\tau))}{d\tau} = \left\langle \frac{dx}{d\tau}, \operatorname{grad}_f x(\tau) \right\rangle = \bar{\lambda}(\tau) \left\langle \frac{dx}{d\tau}, x(\tau) \right\rangle$$

it implies

$$|\lambda(\tau)| \leqslant 2 \frac{\left| \frac{df(x(\tau))}{d\tau} \right|}{\frac{d\|x(\tau)\|^2}{d\tau}}.$$

As $\|x(\tau)\| = b_1\tau^\beta + \cdots$ with $b_1 \in \mathbb{R}_+^*$ and $\beta < 0$ we have, for small enough τ, $|\lambda(\tau)| \leqslant \gamma \frac{\tau^{\rho-1}}{\tau^{2\beta-1}} = \gamma\tau^{\rho-2\beta}$ where γ is a constant. We end the proof be using the characterization of critical value at infinity in [Pa]:

$$\|x(\tau)\|^{1-1/N} \| \operatorname{grad}_f x(\tau) \| = \|x(\tau)\|^{1-1/N} |\lambda(\tau)| \, \|x(\tau)\| \leqslant \gamma\tau^{\rho-\beta/N}$$

As $\rho > 0$ and $\beta < 0$, for all $N > 0$ we have that $\|x(\tau)\|^{1-1/N} \| \operatorname{grad}_f x(\tau)\| \to 0$ as $\tau \to 0$. It implies that the value c (the limit of $f(x(\tau))$ as $\tau \to 0$) is in $\mathcal{B}_\infty$. But as $c \in K$ it is impossible. □

Lemma 5 enables us to get the following result: because of the transversality we can find a vector field tangent to the fibers of f and pointing out the spheres S_R. Integration of such a vector field gives the next lemma (see [HP] Paragraph 2.2 or [Ti] Lemma 1.8).

Lemma 6. *The fibrations $f : f^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ and $f : f^{-1}(K) \longrightarrow K$ are differentiably isomorphic.*

As $\mathring{K}$ is diffeomorphic to $\mathbb{C} \setminus \mathcal{B}$ we have the following fact:

Lemma 7. *The fibrations $f : f^{-1}(\mathring{K}) \longrightarrow \mathring{K}$ and $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ are differentiably isomorphic.*

The following lemma is adapted from [LR]. For completeness we give the proof.

Lemma 8. *Let R, R' with $R \geqslant R'$ be real numbers such that the intersections $f^{-1}(K) \cap S_R$ and $f^{-1}(K) \cap S_{R'}$ are transversal. Let us suppose that $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are fibrations with fibers homotopic to a wedge of ν $(n-1)$-dimensional spheres. Then the fibrations are fiber homotopy equivalent. And for $n \neq 3$ the fibrations are differentiably equivalent.*

Proof. The first part is a consequence of a result of A. Dold [Do, Th. 6.3]. The first fibration is contained in the second. By the result of Dold we only have to prove that if $* \in \partial D_r$ then the inclusion of $F' = f^{-1}(*) \cap B_{R'}$ in $F = f^{-1}(*) \cap B_R$ is a homotopy equivalence. To see this we choose a generic x_0 in $\mathbb{C}^n$ near the origin such that the real function $x \mapsto \|x - x_0\|$ has only non-degenerate critical points of index less than n (see [M1, §7]). Then F is obtained from F' by attaching cells of index less than n.

For $n = 2$ the fibers are homotopic to a wedge of ν circles, then the inclusion of F' in F is a homotopy equivalence. For $n > 2$ the fibers F, F' are simply connected and the morphism $H_i(F') \longrightarrow H_i(F)$ induced by inclusion is an isomorphism. For $i \neq n-1$ this is trivial since F and F' have the homotopy type of a wedge of $(n-1)$-dimensional spheres, and for $i = n-1$ the exact sequence of the pair (F, F') is

$$0 \longrightarrow H_{n-1}(F) \longrightarrow H_{n-1}(F') \longrightarrow H_{n-1}(F, F')$$

with $H_n(F, F') = 0$, $H_{n-1}(F)$ and $H_{n-1}(F')$ free of rank ν, and $H_{n-1}(F, F')$ torsion-free. Then the inclusion of F' in F is a homotopy equivalence.

The second part is based on the h-cobordism theorem. Let $X = f^{-1}(K) \cap B_R \setminus \mathring{B}_{R'}$, then as f has no affine critical point in X (because there is no critical value in K) and f is transversal to $f^{-1}(K) \cap S_R$ and to $f^{-1}(K) \cap S_{R'}$ then, by Ehresmann theorem, $f : X \longrightarrow K$ is a fibration. We denote $F \setminus \mathring{F}'$ by F^*. We get an isomorphism $H_i(\partial F') \longrightarrow H_i(F^*)$ for all i because $H_i(F^*, \partial F') = H_i(F, F') = 0$. For $n = 2$ it implies that F^* is diffeomorphic to a product $[0, 1] \times \partial F'$.

For $n > 3$ we will use the h-cobordism theorem applied to F^* to prove this. We have $\partial F^* = \partial F' \cup \partial F$; $\partial F'$ and ∂F are simply connected: if we look at the function $x \mapsto -\|x - x_0\|$ on $f^{-1}(*)$ for a generic x_0, then $F = f^{-1}(*) \cap B_R$ and $F' = f^{-1}(*) \cap B_{R'}$ are obtained by gluing cells of index more or equal to $n - 1$. So their boundary is simply connected. For a similar reason F^* is simply connected. As we have isomorphisms $H_i(\partial F') \longrightarrow H_i(F^*)$ and both spaces are simply connected then by Hurewicz-Whitehead theorem the inclusion of $\partial F'$ in F^* is a homotopy equivalence.

Now F^*, $\partial F'$, ∂F are simply connected, the inclusion of $\partial F'$ in F^* is a homotopy equivalence and F^* has real dimension $2n - 2 \geqslant 6$. So by the h-cobordism theorem, [M2], F^* is diffeomorphic to the product $[0, 1] \times \partial F'$. Then the fibration $f : X \longrightarrow K$ is differentiably equivalent to the fibration $f : [0, 1] \times (f^{-1}(K) \cap S_{R'}) \longrightarrow K$; so the fibrations $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are differentiably equivalent. □

3. Family of polynomials

Let $(f_t)_{t\in[0,1]}$ be a family of polynomials that verify hypotheses of theorem 1.

Lemma 9 ([HP]). *There exists $R \gg 1$ such that for all $t \in [0,1]$ the affine critical points of f_t are in $\mathring{B}_R$.*

Proof. It is enough to prove it on $[0,\tau]$ with $\tau > 0$. We choose $R \gg 1$ such that all the affine critical points of f_0 are in $\mathring{B}_R$. We denote

$$\phi_t = \frac{\operatorname{grad}_{f_t}}{\| \operatorname{grad}_{f_t} \|} : S_R \longrightarrow S_1.$$

Then $\deg \phi_0 = \mu(0)$. For all $x \in S_R$, $\operatorname{grad}_{f_0} x \neq 0$, and by continuity there exists $\tau > 0$ such that for all $t \in [0,\tau]$ and all $x \in S_R$, $\operatorname{grad}_{f_t} x \neq 0$. Then the maps ϕ_t are homotopic (the homotopy is $\phi : S_R \times [0,\tau] \longrightarrow S_1$ with $\phi(x,t) = \phi_t(x)$). And then $\mu(0) = \deg \phi_0 = \deg \phi_t \leqslant \mu(t)$. If there exists a family $x(t) \in \mathbb{C}^n$ of affine critical points of ϕ_t such that $\|x(t)\| \to +\infty$ as $t \to 0$, then for a sufficiently small t, $x(t) \notin B_R$ and then $\mu(t) > \deg \phi_t$. It contradicts the hypothesis $\mu(0) = \mu(t)$. ☐

Lemma 10. *There exists $r \gg 1$ such that the subset $\big\{(c,t) \in D_r(0) \times [0,1] \mid c \in \mathcal{B}(t)\big\}$ is a braid of $D_r(0) \times [0,1]$.*

It enables us to choose $* \in \partial D_r(0)$ which is a regular value for all f_t, $t \in [0,1]$. In other words if we enumerate $\mathcal{B}(0)$ as $\{c_1(0), \ldots, c_m(0)\}$ then there are continuous functions $c_i : [0,1] \longrightarrow D_r(0)$ such that for $i \neq j$, $c_i(t) \neq c_j(t)$. This enables us to identify $\pi_1(\mathbb{C} \setminus \mathcal{B}(0), *)$ and $\pi_1(\mathbb{C} \setminus \mathcal{B}(1), *)$ by means of the previous braid.

Proof. Let τ be in $[0,1]$ and $c(\tau)$ be a critical value of f_τ, then for all t near τ there exists a critical value $c(t)$ of f_t. It is a hypothesis for critical values at infinity and this fact is well-known for affine critical values as the coefficients of f_t are smooth functions of t, see for example [Br, Prop. 2.1].

Moreover there can not exist critical values that escape at infinity *i.e.* a $\tau \in [0,1]$ such that $|c(t)| \to +\infty$ as $t \to \tau$. For affine critical values it is a consequence of lemma 9 (or we can make the same proof as we now will perform for the critical values at infinity). For $\mathcal{B}_\infty(t)$ let us suppose that there are critical values that escape at infinity. By continuity of the critical values at infinity with respect to t we can suppose that there is a continuous function $c_0(t)$ on $]0,\tau]$ ($\tau > 0$) with $c_0(t) \in \mathcal{B}_\infty(t)$ and $|c(t)| \to +\infty$ as $t \to 0$. By continuity of the critical values at infinity, if $\mathcal{B}_\infty(0) = \{c_1(0), \ldots, c_p(0)\}$ there exist continuous functions $c_i(t)$ on $[0,\tau]$ such that $c_i(t) \in \mathcal{B}_\infty(t)$ for all $i = 1, \ldots, p$.

And for a sufficiently small $t > 0$, $c_0(t) \neq c_i(t)$ $(i = 1, \ldots, p)$ then $\#\mathcal{B}_\infty(0) < \#\mathcal{B}_\infty(t)$ which contradicts the constancy of $\#\mathcal{B}_\infty(t)$.

Finally there can not exist ramification points: suppose that there is a τ such that $c_i(\tau) = c_j(\tau)$ (and $c_i(t), c_j(t)$ are not equal in a neighborhood of τ). Then if $c_i(\tau) \in \mathcal{B}_{\mathit{aff}}(\tau) \setminus \mathcal{B}_\infty(\tau)$ (resp. $\mathcal{B}_\infty(\tau) \setminus \mathcal{B}_{\mathit{aff}}(\tau)$, $\mathcal{B}_\infty(\tau) \cap \mathcal{B}_{\mathit{aff}}(\tau)$) there is a jump in $\#\mathcal{B}_{\mathit{aff}}(t)$ (resp. $\#\mathcal{B}_\infty(t)$, $\#\mathcal{B}(t)$) near τ which is impossible by assumption. □

Let $R_0, K, D_r(0), D_\varepsilon(c)$ be the objects of section 2 for the polynomial $f = f_0$. Moreover we suppose that R_0 is greater than the R obtained in lemma 9.

Lemma 11. *There exists $\tau \in]0,1]$ such that for all $t \in [0,\tau]$ we have the properties:*

- *$c_i(t) \in D_\varepsilon(c_i(0))$, $i = 1, \ldots, m$;*
- *for all $s \in K$, $f_t^{-1}(s)$ intersects S_{R_0} transversally.*

Proof. The first point is just the continuity of the critical values $c_i(t)$. The second point is the continuity of transversality: if the property is false then there exist sequences $t_k \to 0$, $x_k \in S_{R_0}$ and $\lambda_k \in \mathbb{C}$ such that $\operatorname{grad}_{f_{t_k}} x_k = \lambda_k x_k$. We can suppose that (x_k) converges (after extraction of a sub-sequence, if necessary). Then $x_k \to x \in S_{R_0}$, $\operatorname{grad}_{f_{t_k}} x_k \to \operatorname{grad}_{f_0} x$, and $\lambda_k = \langle \operatorname{grad}_{f_{t_k}} x_k \,|\, x_k\rangle / \|x_k\|^2 = \langle \operatorname{grad}_{f_{t_k}} x_k \mid x_k \rangle / {R_0}^2$ converges towards $\lambda \in \mathbb{C}$. Then $\operatorname{grad}_{f_0} x = \lambda x$ and the intersection is non-transversal. □

Lemma 12. *The fibrations $f_0 : f_0^{-1}(K) \cap B_{R_0} \longrightarrow K$ and $f_\tau : f_\tau^{-1}(K) \cap B_{R_0} \longrightarrow K$ are differentiably isomorphic.*

Proof. Let

$$F : \mathbb{C}^n \times [0,1] \longrightarrow \mathbb{C} \times [0,1], \qquad (x,t) \mapsto (f_t(x), t).$$

We want to prove that the fibrations

$$F_0 : \Sigma_0 = F^{-1}(K \times \{0\}) \cap (B_{R_0} \times \{0\}) \longrightarrow K, \qquad (x,0) \mapsto f_0(x)$$

and

$$F_\tau : \Sigma_\tau = F^{-1}(K \times \{\tau\}) \cap (B_{R_0} \times \{\tau\}) \longrightarrow K, \qquad (x,\tau) \mapsto f_\tau(x)$$

are differentiably isomorphic. Let denote $[0,\tau]$ by I. Then F has maximal rank on $F^{-1}(K \times I) \cap (\mathring{B}_{R_0} \times I)$ and on the boundary $F^{-1}(K \times I) \cap (S_{R_0} \times I)$. By Ehresmann theorem $F : F^{-1}(K \times I) \cap (B_{R_0} \times I) \longrightarrow K \times I$ is a fibration.

As in [HP] we build a vector field that gives us a diffeomorphism between the two fibrations F_0 and F_τ. Moreover it provides a control of the diffeomorphism near S_{R_0} that we will need later. Let $0 < \eta \ll 1$ be a real number. We build a vector field v_1:

- which is defined on $F^{-1}(K \times I) \cap (\cup_{R_0-2\eta<R<R_0} S_R \times I)$,
- such that $d_z F.v_1(z) = (0,1)$ for all z,
- and such that $v_1(z)$ is tangent to $S_R \times I$ for $z \in S_R \times I$, $R_0 - 2\eta < R < R_0$.

This is possible because F is a fibration on $F^{-1}(K \times I) \cap (B_{R_0} \times I)$. On the set $F^{-1}(K \times I) \cap (\mathring{B}_{R_0-\eta} \times I)$ we build a second vector field v_2 such that $d_z F.v_2(z) = (0,1)$.

By gluing these vector fields v_1 and v_2 by a partition of unity and by integrating the corresponding vector field we obtain integral curves

$$p_z : [0,1] \longrightarrow F^{-1}(K \times I) \cap B_{R_0} \times I$$

such that $p_z(0) = z \in \Sigma_0$ and $p_z(\tau) \in \Sigma_\tau$. It induces a diffeomorphism $\Phi : \Sigma_0 \longrightarrow \Sigma_\tau$ such that $F_0 = F_\tau \circ \Phi$; that makes the fibrations isomorphic. □

Proof of theorem 1. It is sufficient to prove the theorem for a family (f_t) parameterized by t in an interval $[0,\tau]$ for a small $\tau > 0$. We choose τ as in lemma 11. By lemma 7, $f_0 : f^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_0 : f_0^{-1}(\mathring{K}) \longrightarrow \mathring{K}$ are differentiably isomorphic fibrations. Then by lemma 6, the fibration $f_0 : f_0^{-1}(K) \longrightarrow K$ is differentiably isomorphic to $f_0 : f_0^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ which is, by lemma 12 differentiably isomorphic to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$.

By continuity of transversality (lemma 11) $f_\tau^{-1}(K)$ has transversal intersection with S_{R_0}. Lemma 5 applied to f_τ gives us a large real number R, such that $f_\tau^{-1}(K)$ intersects S_R transversally, R may be much more greater than R_0. The fibration $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ is fiber homotopy equivalent to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_R \longrightarrow K$: it is the first part of lemma 8 because the fiber $f_\tau^{-1}(*) \cap \mathring{B}_{R_0}$ is homotopic to a wedge of $\mu(0) + \lambda(0)$ spheres and the fiber $f_\tau^{-1}(*) \cap \mathring{B}_R$ is homotopic to a wedge of $\mu(\tau) + \lambda(\tau)$ spheres; as $\mu(0) + \lambda(0) = \mu(\tau) + \lambda(\tau)$ we get the desired conclusion. Moreover for $n \neq 3$ by the second part of lemma 8 the fibrations are differentiably isomorphic.

By applying lemmas 6 and 7 to f_τ, the fibration $f_\tau : f_\tau^{-1}(\mathring{K}) \cap \mathring{B}_R \longrightarrow \mathring{K}$ is differentiably isomorphic to $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$. As a conclusion the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic □

4. Around affine singularities

We now work with $t \in [0,1]$. We suppose in this paragraph that the critical values $\mathcal{B}(t)$ depend analytically on $t \in [0,1]$. This enables us to construct a diffeomorphism χ such that:

- $\chi : \mathbb{C} \times [0,1] \longrightarrow \mathbb{C} \times [0,1]$,
- $\chi(x,t) = (\chi_t(x), t)$,
- $\chi_0 = \mathrm{id}$,
- $\chi_t(\mathcal{B}(t)) = \mathcal{B}(0)$.

We denote χ_1 by Ψ, so that $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ verifies $\Psi(\mathcal{B}(1)) = \mathcal{B}(0)$. Moreover we can suppose that χ_t is equal to id on $\mathbb{C} \setminus D_r(0)$ because all the critical values are in $D_r(0)$. Finally χ defines a vector field w of $\mathbb{C} \times [0,1]$ by $\frac{\partial \chi}{\partial t}$.

We need a non-splitting result of the affine singularities, this principle has been proved by C. Has Bey ([HB], $n = 2$) and by F. Lazzeri ([La], for all n).

Lemma 13. *Let $x(\tau)$ be an affine singular point of f_τ and let U_τ be an open neighborhood of $x(\tau)$ in $\mathbb{C}^n$ such that $x(\tau)$ is the only affine singular point of f_τ in U_τ. Suppose that for all t closed to τ, the restriction of f_t to U_τ has only one critical value. Then for all t sufficiently closed to τ, there is one, and only one, affine singular point of f_t contained in U_τ.*

This lemma is a local lemma; it enables to enumerate the singularities: if we denote the affine singular points of f_0 by $\{x_i(0)\}_{i \in J}$ then there are continuous functions $x_i : [0,1] \longrightarrow \mathbb{C}^n$ such that $\{x_i(t)\}_{i \in J}$ is the set of affine singularities of f_t. Let us notice that there can be two distinct singular points of f_t with the same critical value.

We suppose

- that (f_t) verifies the hypotheses of theorem 1,
- that $n \neq 3$,
- and $\mathcal{B}(t)$ depends analytically on t.

This and lemma 13 imply that for all $t \in [0,1]$ the local Milnor number of f_t at $x(t)$ is equal to the local Milnor number of f_0 at $x(0)$. The improved version of Lê-Ramanujam theorem by J.G. Timourian [Tm] for a family of germs with constant local Milnor number proves that (f_t) is locally a product family.

Theorem 14 (Lê-Ramanujam-Timourian)**.** *Let $x(t)$ be a singular point of f_t. There exist U_t, V_t neighborhoods of $x(t)$, $f_t(x(t))$ respectively* *and a homeomorphism Ω^{in} such that if $U = \bigcup_{t\in[0,1]} U_t \times \{t\}$ and $V = \bigcup_{t\in[0,1]} V_t \times \{t\}$ the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{\Omega^{\text{in}}} & U_0 \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times \mathrm{id}} \\ V & \xrightarrow[\chi]{} & V_0 \times [0,1]. \end{array}$$

In particular it proves that the polynomials f_0 and f_1 are locally topologically equivalent: we get a homeomorphism Φ_{in} such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{\Phi_{\text{in}}} & U_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ V_1 & \xrightarrow[\Psi]{} & V_0. \end{array}$$

By lemma 9 we know that for all $t \in [0,1]$, $\mathcal{B}(t) \subset D_r(0)$. Now we redefine the radius R_0 and R_1 of section 2. By continuity of transversality and compactness of $[0,1]$ we choose R_1 such that

$$\forall c \in \mathcal{B}(0) \quad \forall R \geqslant R_1 \quad f_0^{-1}(c) \pitchfork S_R \quad \text{and} \quad \forall t \in [0,1] \quad \forall c \in \mathcal{B}(t) \quad f_t^{-1}(c) \pitchfork S_{R_1}.$$

For a sufficiently small ε we denote

$$K(0) = D_r(0) \setminus \bigcup_{c\in\mathcal{B}_\infty(0)} \mathring{D}_\varepsilon(c), \qquad K(t) = \chi_t^{-1}(K(0))$$

and we choose $R_0 \geqslant R_1$ such that

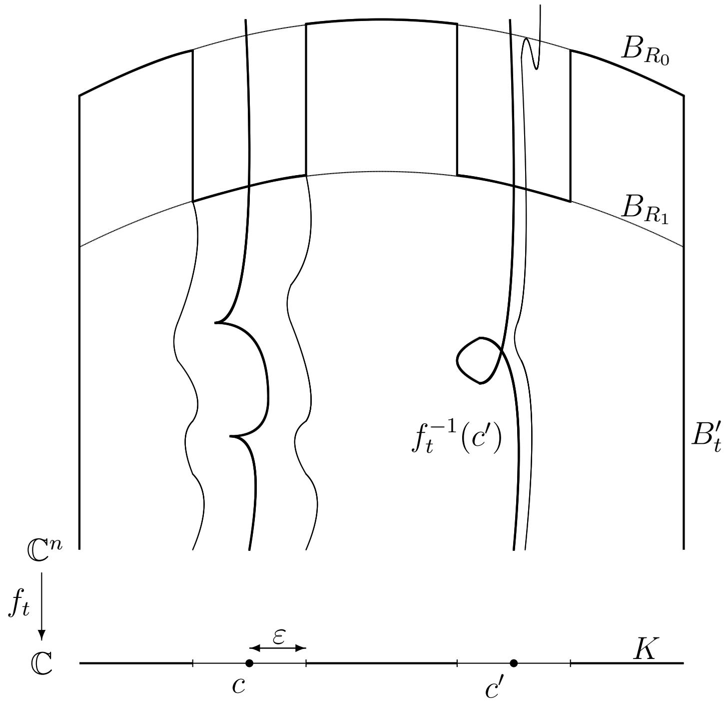
$$\forall s \in K(0) \quad \forall R \geqslant R_0 \quad f_0^{-1}(s) \pitchfork S_R \quad \text{and} \quad \forall t \in [0,1] \quad \forall s \in K(t) \quad f_t^{-1}(s) \pitchfork S_{R_0}.$$

We denote

$$B'_t = \big(f_t^{-1}(D_r(0)) \cap B_{R_1}\big) \cup \big(f_t^{-1}(K(t)) \cap B_{R_0}\big), \quad t \in [0,1].$$

Lemma 15. *There exists a homeomorphism Φ such that we have a commutative diagram:*

$$\begin{array}{ccc} B'_1 & \xrightarrow{\Phi} & B'_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ D_r(0) & \xrightarrow[\Psi]{} & D_r(0). \end{array}$$



Proof. We denote by U'_t a neighborhood of $x(t)$ such that $\bar{U}'_t \subset U_t$. We denote by $\mathcal{U}_t$ (resp. $\mathcal{U}'_t$), the union (on the affine singular points of f_t) of the U_t (resp. U'_t). We set

$$B''_t = B'_t \setminus \mathcal{U}'_t, \quad t \in [0,1].$$

We can extend the homeomorphism Φ of lemma 12 to $\Phi_{\text{out}} : B''_1 \longrightarrow B''_0$. We just have to extend the vector field of lemma 12 to a new vector field denoted by v' such that

- v' is tangent to $\partial\mathcal{U}'_t$,
- v' is tangent to $S_{R_1} \times [0,1]$ on $F^{-1}(D_r(0) \setminus K(t) \times \{t\})$ for all $t \in [0,1]$,
- v' is tangent to $S_{R_0} \times [0,1]$ on $F^{-1}(K(t) \times \{t\})$ for all t.
- $d_z F.v'(z) = w(F(z))$ for all $z \in \bigcup_{t\in[0,1]} B''_t \times \{t\}$, which means that Φ_{out} respects the fibrations (w is defined by $\frac{\partial \chi}{\partial t}$).

If we set $B'' = \bigcup_{t\in[0,1]} B''_t \times \{t\}$ the integration of v' gives Ω^{out} and Φ_{out} such that:

$$\begin{array}{ccc} B'' & \xrightarrow{\Omega^{\mathrm{out}}} & B''_0 \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times \mathrm{id}} \\ D_r(0)\times[0,1] & \xrightarrow[\chi]{} & D_r(0)\times[0,1], \end{array} \qquad \begin{array}{ccc} B''_1 & \xrightarrow{\Phi_{\mathrm{out}}} & B''_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ D_r(0) & \xrightarrow[\Psi]{} & D_r(0). \end{array}$$

We now explain how to glue Φ_{in} and Φ_{out} together. We can suppose that there exist spheres S_t centered at the singularities $x(t)$ such that if $S = \bigcup_{t\in[0,1]} S_t \times \{t\}$ then we have $\Omega^{\mathrm{in}} : S \longrightarrow S_0 \times [0,1]$ and $\Omega^{\mathrm{out}} : S \longrightarrow S_0 \times [0,1]$. It defines $\Omega^{\mathrm{in}}_t : S_t \longrightarrow S_0$ and $\Omega^{\mathrm{out}}_t : S_t \longrightarrow S_0$. On S_1 we have $\Omega^{\mathrm{in}}_1 = \Phi_{\mathrm{in}}$ and $\Omega^{\mathrm{out}}_1 = \Phi_{\mathrm{out}}$.

Now we define

$$\Theta_t : S_1 \longrightarrow S_0, \quad \Theta_t = \Omega^{\mathrm{in}}_t \circ (\Omega^{\mathrm{out}}_t)^{-1} \circ \Phi_{\mathrm{out}}.$$

Then $\Theta_0 = \Phi_{\mathrm{out}}$ and $\Theta_1 = \Phi_{\mathrm{in}}$. On a set homeomorphic to $S \times [0,1]$ included in $\bigcup_{t\in[0,1]} U_t \setminus U'_t$ we glue Φ_{in} to Φ_{out}, moreover this gluing respects the fibrations f_0 and f_1. We end by doing this construction for all affine singular points. $\square$

Proof of theorem 3. In the hypotheses of this theorem we supposed that there is no critical value at infinity. In order to apply the results of this section we have to prove that affine critical values are analytic functions of t. Let $c(0) \in \mathcal{B}_{\mathit{aff}}(0)$, by lemma 10 it defines a continuous function $c : [0,1] \longrightarrow \mathbb{C}$. The set $\mathcal{C} = \big\{(c(t),t) \mid t \in [0,1]\big\}$ is a real algebraic subset of $\mathbb{C}\times[0,1]$ as all affine critical points are contained in B_{R_0} (lemma 9). In fact there is a polynomial $P \in \mathbb{C}[x,t]$ such that $\mathcal{C}$ is equal to $(P=0)\cap(\mathbb{C}\times[0,1])$. Because the set of critical values is a braid of $\mathbb{C}\times[0,1]$ (lemma 10) then $c : [0,1] \longrightarrow \mathbb{C}$ is an analytic function.

If we suppose that $\mathcal{B}_\infty(t) = \varnothing$ for all $t \in [0,1]$ then by lemma 6 we can extend $\Phi : B'_1 \longrightarrow B'_0$ to $\Phi : f_1^{-1}(D_r(0)) \longrightarrow f_0^{-1}(D_r(0))$. And as $\mathcal{B}(t) \subset D_r(0)$ by a lemma similar to lemma 7 we can extend the homeomorphism to the whole space. $\square$

Remark. We can improve the end of the proof of lemma 15 in order to get a trivialization of the whole family, that is to say $(f_t)_{t\in[0,1]}$ is topologically a product family. For each $t\in[0,1]$ we thicken the sphere S_t in a set $S_t \times [0,1]$. We parameterize this interval $[0,1]$ by s. Let

$$\Lambda : S\times[0,1] \longrightarrow S_0\times[0,1]\times[0,1], \quad \Lambda(x,t,s) \mapsto (\Lambda_{t,s}(x),t,s)$$

where $\Lambda_{t,s}$ is a map defined by

$$\Lambda_{t,s} : S_t \longrightarrow S_0, \quad \Lambda_{t,s} = \Omega^{\text{in}}_{s\times t} \circ (\Omega^{\text{out}}_{s\times t})^{-1} \circ \Omega^{\text{out}}_{t}.$$

By fixing $s = 0$ the map Λ can be identified with Ω^{out} and for $s = 1$ it can be identified with Ω^{in}. So we are able to glue together the trivializations in order to get a homeomorphism Ω with a commutative diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\Omega} & B'_0 \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times\mathrm{id}} \\ D_r(0)\times[0,1] & \xrightarrow[\chi]{} & D_r(0)\times[0,1], \end{array}$$

where $B' = \bigcup_{t\in[0,1]} B'_t \times \{t\}$. Now if $\mathcal{B}_\infty(t)$ is empty for all $t \in [0,1]$, then we can extend Ω in order to get:

$$\begin{array}{ccc} \mathbb{C}^n\times[0,1] & \xrightarrow{\Omega} & \mathbb{C}^n\times[0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times\mathrm{id}} \\ \mathbb{C}\times[0,1] & \xrightarrow[\chi]{} & \mathbb{C}\times[0,1]. \end{array}$$

5. Polynomials in two variables

We set $n = 2$. We recall a result of L. Fourrier [Fo]. Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ with set of critical values at infinity $\mathcal{B}_\infty$. Let $* \notin \mathcal{B}$ and $Z = f^{-1}(*) \cup \bigcup_{c\in\mathcal{B}_\infty} f^{-1}(c)$. The *total link of f* is $L_f = Z \cap S_R$ for a sufficiently large R.

To f we associate a resolution $\phi : \Sigma \longrightarrow \mathbb{P}^1$,

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{P}^2 & \xleftarrow{\pi} & \Sigma \\ {\scriptstyle f}\downarrow & & {\scriptstyle \tilde f}\downarrow & \swarrow{\scriptstyle \phi} & \\ \mathbb{C} & \longrightarrow & \mathbb{P}^1 & & \end{array}$$

where $\tilde f$ is the map coming from the homogenization of f; π is the minimal blow-up of some points on the line at infinity $\mathcal{L}_\infty$ of $\mathbb{P}^2$ in order to obtain a well-defined morphism $\phi : \Sigma \longrightarrow \mathbb{P}^1$. The components of the divisor $\pi^{-1}(\mathcal{L}_\infty)$ on which ϕ is surjective are the *dicritical components.* For each dicritical component D we have a branched covering $\phi : D \longrightarrow \mathbb{P}^1$. If the union of dicritical components is D_{dic} we then have the restriction $\phi_{\text{dic}} : D_{\text{dic}} \longrightarrow \mathbb{P}^1$ of ϕ. The 0-*monodromy representation* is the representation

$$\pi_1(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \operatorname{Aut}\left(\phi^{-1}_{\text{dic}}(*)\right).$$

The set $\phi_{\mathrm{dic}}^{-1}(*)$ is in bijection with the boundary components of $f^{-1}(*)$. Then the 0-monodromy representation can be seen as the action of $\pi_1(\mathbb{C} \setminus \mathcal{B})$ on the boundary components of $f^{-1}(*)$.

Theorem 16 (Fourrier)**.** *Let f, g be complex polynomials in two variables with equivalent* 0*-monodromy representations and equivalent total links. Then there exist compact sets C, C' of $\mathbb{C}^2$ and homeomorphisms Φ_∞ and Ψ_∞ that make the diagram commute:*

$$\begin{array}{ccc} \mathbb{C}^2 \setminus C & \xrightarrow{\Phi_\infty} & \mathbb{C}^2 \setminus C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\Psi_\infty]{} & \mathbb{C}. \end{array}$$

Let $f_t : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that the coefficients of this family are algebraic in t. We suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{\mathit{aff}}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t \in [0,1]$. We also suppose the $\deg f_t$ does not depend on t. For our family (f_t), by theorem 1 we know that the geometric monodromy representations are all equivalent, then they act similarly on the boundary components of $f_t^{-1}(*)$. It implies that all the 0-monodromy representations of (f_t) are equivalent. Moreover if we suppose that for any $t, t' \in [0,1]$ the total links L_{f_t} and $L_{f_{t'}}$ are equivalent, then by theorem 16 the polynomials f_t and $f_{t'}$ are topologically equivalent out of some compact sets of $\mathbb{C}^2$. We need a result a bit stronger which can be proved by similar arguments than in [Fo] and we will omit the proof:

Lemma 17. *Let $(f_t)_{t\in[0,1]}$ be a polynomial family such that the coefficients are algebraic functions of t. We suppose that the* 0*-monodromy representations and the total links are all equivalent. Then there exist compact sets $C(t)$ of $\mathbb{C}^2$ and a homeomorphism Ω^∞ such that if $\mathcal{C} = \bigcup_{t\in[0,1]} C(t) \times \{t\}$ we have a commutative diagram:*

$$\begin{array}{ccc} \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} & \xrightarrow{\Omega^\infty} & \big(\mathbb{C}^2 \setminus C(0)\big) \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0 \times \mathrm{id}} \\ \mathbb{C} \times [0,1] & \xrightarrow[\chi]{} & \mathbb{C} \times [0,1]. \end{array}$$

We now prove a strong version of the continuity of critical values.

Lemma 18. *The critical values are analytic functions of t. Moreover for $c(t) \in \mathcal{B}(t)$, the integers $\mu_{c(t)}$ and $\lambda_{c(t)}$ do not depend on $t \in [0,1]$.*

Proof. For affine critical values, refer to the proof of theorem 3. The constancy of $\mu_{c(t)}$ is a consequence of lemma 9 and lemma 13. For critical values at infinity we need a result of [Ha] and [HP] that enables to calculate critical values and Milnor numbers at infinity. As $\deg f_t$ is constant we can suppose that this degree is $\deg_y f_t$. Let denote $\Delta(x,s,t)$ the discriminant $\mathrm{Disc}_y(f_t(x,y)-s)$ with respect to y. We write

$$\Delta(x,s,t) = q_1(s,t)x^{k(t)} + q_2(s,t)x^{k(t)-1} + \cdots$$

First of all Δ has constant degree $k(t)$ in x because $k(t) = \mu(t)+\lambda(t)+\deg f_t - 1$ (see [HP]). Secondly by [Ha] we have

$$\mathcal{B}_\infty(t) = \big\{ s \mid q_1(s,t) = 0 \big\}$$

then we see that critical values at infinity depend continuously on t and that critical values at infinity are a real algebraic subset of $\mathbb{C}\times[0,1]$. For the analyticity we end as in the proof of theorem 3. Finally, for a fixed t, we have that $\lambda_c = k(t) - \deg_x \Delta(x,c,t)$. In other words $q_i(c,t)$ is zero for $i = 1,\ldots,\lambda_c$ and non-zero for $i = \lambda_c+1$. For $c(t) \in \mathcal{B}_\infty(t)$ we now prove that $\lambda_{c(t)}$ is constant. The former formula proves that $\lambda_{c(t)}$ is constant except for finitely many $\tau \in [0,1]$ for which $\lambda_{c(\tau)} \geqslant \lambda_{c(t)}$. But if $\lambda_{c(\tau)} > \lambda_{c(t)}$ then $\lambda(\tau) = \sum_{c\in\mathcal{B}_\infty(\tau)} \lambda_c > \sum_{c\in\mathcal{B}_\infty(t)} \lambda_c = \lambda(t)$ which contradicts the hypotheses. □

To apply lemma 17 we need to prove:

Lemma 19. *For any $t,t' \in [0,1]$ the total links L_{f_t} and $L_{f'_t}$ are equivalent.*

Proof. The problem is similar to the one of [LR] and to lemma 8. Let $c(t) \in \mathcal{B}_\infty(t) \cup \{*\}$. As in lemma 15 we have $R_1 \gg 1$ such that $f_0^{-1}(c(0)) \cap S_{R_1}$ is the link at infinity of $f_0^{-1}(c(0))$. Moreover by lemma 15 we know that the link at infinity $f_0^{-1}(c(0)) \cap S_{R_1}$ is equivalent to the link $f_1^{-1}(c(1)) \cap S_{R_1}$. But $f_1^{-1}(c(1)) \cap S_{R_1}$ is not necessarily the link at infinity for $f_1^{-1}(c(1))$.

We now prove this fact; let denote $c = c(1)$. Let $R_2 \geqslant R_1$ such that for all $R \geqslant R_2$, $f_1^{-1}(c) \pitchfork S_R$, then $f_1^{-1}(c) \cap S_{R_2}$ is the link at infinity of $f_1^{-1}(c)$. We choose η, $0 < \eta \ll 1$ such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_1} and S_{R_2} and such that $f_1^{-1}(\partial D_\eta(c))$ has transversal intersection with all S_R, $R \in [R_1, R_2]$. Notice that η is much smaller than the ε of the former paragraphs and that $f_1^{-1}(s) \cap S_{R_2}$ is *not* the link at infinity of $f_1^{-1}(s)$ for $s \in \partial D_\eta(c)$. We fix R_0 smaller than R_1 such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_0}. We denote $f_1^{-1}(D_\eta(c)) \cap B_{R_i} \setminus \mathring{B}_{R_0}$ by $\mathcal{A}_i$, $i = 1,2$.

The proof is now similar to the one of lemma 8. Let A_1 and A_2 be connected components of $\mathcal{A}_1$ and $\mathcal{A}_2$ with $A_1 \subset A_2$. By Ehresmann theorem, we have fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$, $f_1 : A_2 \longrightarrow D_\eta(c)$.

From one hand $f_1^{-1}(c) \cap B_{R_1}$ is diffeomorphic to $f_0^{-1}(c(0)) \cap B_{R_1}$. So by Suzuki formula (see [HL]) $f_1^{-1}(c) \cap B_{R_1}$ has Euler characteristic $1 - \mu - \lambda + \mu_{c(0)} + \lambda_{c(0)}$. From the other hand $f_1^{-1}(c) \cap B_{R_2}$ has Euler characteristic $1 - \mu - \lambda + \mu_{c(1)} + \lambda_{c(1)}$ by Suzuki formula. By lemma 18 we have that $\mu_{c(0)} + \lambda_{c(0)} = \mu_{c(1)} + \lambda_{c(1)}$, with $c = c(1)$. So the fiber $f_1^{-1}(c) \cap B_{R_1}$ and $f_1^{-1}(c) \cap B_{R_2}$ have the same Euler characteristic. As the number of connected components of $f_1^{-1}(c) \cap B_R$ is constant for $R \in [R_1, R_2]$ we have that $f_1^{-1}(c) \cap B_{R_1}$ and $f_1^{-1}(c) \cap B_{R_2}$ are homotopic. It implies that the fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$ and $f_1 : A_2 \longrightarrow D_\eta(c)$ are fiber homotopy equivalent, and even more are diffeomorphic.

It provides a diffeomorphism $\Xi : A_1 \cap S_{R_1} = A_2 \cap S_{R_1} \longrightarrow A_2 \cap S_{R_2}$ and we can suppose that $\Xi(f_1^{-1}(c) \cap A_1 \cap S_{R_1})$ is equal to $f_1^{-1}(c) \cap A_2 \cap S_{R_2}$. By doing this for all connected components of $\mathcal{A}_1$, $\mathcal{A}_2$, for all values $c \in \mathcal{B}_\infty(1) \cup \{*\}$ and by extending Ξ to the whole spheres we get a diffeomorphism $\Xi : S_{R_1} \longrightarrow S_{R_2}$ such that $\Xi(f_1^{-1}(c) \cap S_{R_1}) = f_1^{-1}(c) \cap S_{R_2}$ for all $c \in \mathcal{B}_\infty(1) \cup \{*\}$. Then the total links for f_0 and f_1 are equivalent. □

Proof of theorem 4. By lemma 17 we have a trivialization $\Omega^\infty : \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} \longrightarrow (\mathbb{C}^2 \setminus C(0)) \times [0,1]$. We can choose the R_1 (before lemma 15) such that $\mathring{C}(t) \subset B_{R_1}$. And then the proof of lemma 15 gives us an $\Omega^{\text{out}} : \bigcup_{t \in [0,1]} B''(t) \times \{t\} \longrightarrow B''(0) \times [0,1]$. By gluing Ω^{out} and Ω^∞ as in the proof of lemma 15, we obtain $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathbb{C} & \xrightarrow[\Psi]{} & \mathbb{C}. \end{array}$$

Then f_0 and f_1 are topologically equivalent. □

Remark. As in the remark after the proof of theorem 3, we can glue Ω^{out} and Ω^∞ in order to get a topologically product family.

6. Continuity of the critical values at infinity

We now give a proof of the second part of lemma 2 in the introduction. The first part has been proven in lemma 18.

Lemma 20. *Let $(f_t)_{t \in [0,1]}$ be a family of polynomials such that the coefficients are polynomials in t. We suppose that:*

- *the total affine Milnor number $\mu(t)$ is constant;*
- *the degree $\deg f_t$ is constant;*

- *the set of critical points at infinity $\Sigma(t)$ is finite and does not vary: $\Sigma(t) = \Sigma$;*
- *for all $(x:0) \in \Sigma$, the generic Milnor number $\nu_{\check{x}}(t)$ is independent of t.*

Then the critical values at infinity depend continuously on t, i.e. *if $c(t_0) \in \mathcal{B}_\infty(t_0)$ then for all t near t_0 there exists $c(t)$ near $c(t_0)$ such that $c(t) \in \mathcal{B}_\infty(t)$.*

Let f be a polynomial. For $x \in \mathbb{C}^n$ we have $(x:1)$ in $\mathbb{P}^n$ and if $x_n \neq 0$ we divide x by x_n to obtain local coordinates at infinity $(\check{x}', x_0)$. The following lemma explains the link between the critical points of f and those of F_c. It uses Euler relation for the homogeneous polynomial part of f of degree d.

Lemma 21.

- *F_c has a critical point $(\check{x}', x_0)$ with $x_0 \neq 0$ of critical value 0 if and only if f has a critical point x with critical value c.*
- *F_c has a critical point $(\check{x}', 0)$ of critical value 0 if and only if $(x:0) \in \Sigma$.*

Proof of lemma 20. We suppose that critical values at infinity are *not* continuous functions of t. Then there exists (t_0, c_0) such that $c_0 \in \mathcal{B}_\infty(t_0)$ and for all (t,c) in a neighborhood of (t_0, c_0), we have $c \notin \mathcal{B}_\infty(t)$. Let P be the point of irregularity at infinity for (t_0, c_0). Then $\mu_P(F_{t_0,c_0}) > \mu_P(F_{t_0,c})$ $(c \neq c_0)$ by definition of $c_0 \in \mathcal{B}_\infty(t_0)$ and by semi-continuity of the local Milnor number at P we have $\nu_P(t_0) = \mu_P(F_{t_0,c}) \geqslant \mu_P(F_{t,c}) = \nu_P(t)$, $(t,c) \neq (t_0, c_0)$.

We consider t as a complex parameter. By continuity of the critical points and by conservation of the Milnor number for $(t,c) \neq (t_0, c_0)$ we have critical points $M(t,c)$ near P of $F_{t,c}$ that are not equal to P. This fact uses that $\deg f_t$ is a constant, in order to prove that $F_{t,c}$ depends continuously on t.

Let denote by V' the algebraic variety of $\mathbb{C}^3 \times \mathbb{C}^n$ defined by $(t,c,s,x) \in V'$ if and only if $F_{t,c}$ has a critical point x with critical value s (the equations are $\operatorname{grad} F_{t,c}(x) = 0, F_{t,c}(x) = s$). If $\mu_P(F_{t,c}) > 0$ for a generic (t,c) then $\left\{(t,c,0,P) \mid (t,c) \in \mathbb{C}^2\right\}$ is a subvariety of V'. We define V to be the closure of V' minus this subvariety. Then for a generic (t,c), $(t,c,0,P) \notin V$. We call $\pi : \mathbb{C}^3 \times \mathbb{C}^n \longrightarrow \mathbb{C}^3$ the projection on the first factor. We set $W = \pi(V)$. Then W is locally an algebraic variety around $(t_0, c_0, 0)$. For each (t,c) there is a non-zero finite number of values s such that $(t,c,s) \in W$. So W is locally an equi-dimensional variety of codimension 1. Then it is a germ of hypersurface of $\mathbb{C}^3$. Let $R(t,c,s)$ be the polynomial that defines W locally.

We set $Q(t,c) = R(t,c,0)$. As $Q(t_0,c_0) = 0$ then in all neighborhoods of (t_0,c_0) there exists $(t,c) \neq (t_0,c_0)$ such that $Q(t,c) = 0$. Moreover there are solutions for t a real number near t_0 and we now suppose that t is a real parameter.

Then for $(t,c) \neq (t_0,c_0)$ we have that: $Q(t,c) = 0$ if and only if $F_{t,c}$ has a critical point $M(t,c) \neq P$ with critical value 0. The point $M(t,c)$ is not equal to P because for $t \neq t_0$, $(t,c,0,P) \notin V$: it uses that $c \notin \mathcal{B}_\infty(t)$ for $t \neq t_0$, and that $\nu_P(t) = \nu_P(t_0)$. Let us notice that $M(t,c) \to P$ as $(t,c) \to (t_0,c_0)$.

We end the proof be studying the different cases:

- if we have $M(t,c)$ in $\mathcal{H}_\infty$ (of equation $(x_0 = 0)$) then $M(t,c) \in \Sigma$ which provides a contradiction because then it is equal to P;
- if we have points $M(t,c)$, not in $\mathcal{H}_\infty$, with $t \neq t_0$ then there are affine critical points $M'(t,c)$ of f_t (lemma 21), and as $M(t,c)$ tends towards P (as (t,c) tends towards (t_0,c_0)) we have that $M'(t,c)$ escapes at infinity. It contradicts the fact that the critical points of f_t are bounded (lemma 9).
- if we have points $M(t_0,c)$, not in $\mathcal{H}_\infty$, then there are infinitely many affine critical points for f_{t_0}, which is impossible since the singularities of f_{t_0} are isolated.

□

7. Examples

Example 1. Let $f_t = x(x^2y + tx + 1)$. Then $\mathcal{B}_{\mathit{aff}}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, $\lambda(t) = 1$ and $\deg f_t = 4$. Then by theorem 4, f_0 and f_1 are topologically equivalent. These are examples of polynomials that are topologically but not algebraically equivalent, see [Bo].

Example 2. Let $f_t = (x+t)(xy+1)$. Then f_0 and f_1 are not topologically equivalent. One has $\mathcal{B}_\infty(t) = \varnothing$, $\mathcal{B}_{\mathit{aff}}(t) = \{0,t\}$ for $t \neq 0$, but $\mathcal{B}_\infty(0) = \{0\}$, $\mathcal{B}_{\mathit{aff}}(0) = \varnothing$. In fact the two affine critical points for f_t "escape at infinity" as t tends towards 0.

Example 3. Let $f_t = x\big(x(y+tx^2)+1\big)$. Then f_0 is topologically equivalent to f_1. We have for all $t \in [0,1]$, $\mathcal{B}_{\mathit{aff}}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, and $\lambda(t) = 1$, but $\deg f_t = 4$ for $t \neq 0$ while $\deg f_0 = 3$.

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Non reality and non connectivity of complex polynomials

NON REALITY AND NON CONNECTIVITY OF COMPLEX POLYNOMIALS

ARNAUD BODIN

Abstract. Using the same method we provide negative answers to the following questions: Is it possible to find real equations for complex polynomials in two variables up to topological equivalence (Lee Rudolph)? Can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials?

1. Introduction

Two polynomials $f, g \in \mathbb{C}[x,y]$ are *topologically equivalent*, and we will denote $f \approx g$, if there exist homeomorphisms $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ and $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $g \circ \Phi = \Psi \circ f$. They are *algebraically equivalent*, and we will denote $f \sim g$, if we have $\Phi \in \operatorname{Aut} \mathbb{C}^2$ and $\Psi = \mathrm{id}$.

It is always possible to find real equations for germs of plane curves up to topological equivalence. In fact the proof is as follows: the topological type of a germ of plane curve $(C, 0)$ is determined by the characteristic pairs of the Puiseux expansions of the irreducible branches and by the intersection multiplicities between these branches. Then we can choose the coefficients of the Puiseux expansions in $\mathbb{R}$ (even in $\mathbb{Z}$). Now it is possible (see [7], appendix to chapter 1) to find a polynomial $f \in \mathbb{R}[x,y]$ (even in $\mathbb{Z}[x,y]$) such that the germ $(f = 0, 0)$ is equivalent to the germ $(C, 0)$.

This property has been widely used by N. A'Campo and others (see [1] for example) in the theory of divides. Lee Rudolph asked the question whether it is true for polynomials [10]. We give a negative answer:

Theorem A. *Up to topological equivalence it is not always possible to find real equations for complex polynomials.*

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We now deal with another problem. In [5] we proved that a family of polynomials with some constant numerical data are all topologically equivalent. More precisely for a polynomial let $\mathfrak{m} = (\mu, \#\mathcal{B}_{\mathit{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$ be the multi-integer respectively composed of the affine Milnor number, the number of affine critical values, the Milnor number at infinity, the number of critical values at infinity, the number of critical values (with $\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty$). Then we have a global version of the Lê-Ramanujam μ-constant theorem:

Theorem ([5]). *Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials in two variables whose coefficients are polynomials in t. Suppose that the multi-integer $\mathfrak{m}(t)$ and the degree $\deg f_t$ do not depend on $t \in [0,1]$. Then the polynomials f_0 and f_1 are topologically equivalent.*

It is true that two topologically equivalent polynomials have the same multi-integers $\mathfrak{m}$. A natural question is: can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials ?

Theorem B. *There exist two topologically equivalent polynomials f_0, f_1 that cannot be connected by a family of equivalent polynomials. That means that for each continuous family $(f_t)_{t\in[0,1]}$ there exists a $\tau \in]0,1[$ such that f_τ is not topologically equivalent to f_0.*

It can be noticed that the answer is positive for algebraic equivalence. Two algebraically equivalent polynomials can be connected by algebraically equivalent polynomials since $\operatorname{Aut}\mathbb{C}^2$ is connected by Jung's theorem.

Such kinds of problems have been studied by V. Kharlamov and V. Kulikov in [9] for cuspidal projective curves. They give two complex conjugate projective curves that are not isotopic. The example with lowest degree has degree 825. In [2], E. Artal, J. Carmona and L. Cogolludo give examples of projective curves C, C' of degree 6 that have conjugate equations in $\mathbb{Q}(\sqrt{2})$ but the pairs $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic by an orientation-preserving homeomorphism.

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The method used in this note is based on the relationship between topological and algebraic equivalence: we set a family $(f_s)_{s\in\mathbb{C}}$ of polynomials such that $(f_s = 0)$ is a line arrangement in $\mathbb{C}^2$. One of the line depends on a parameter $s \in \mathbb{C}$. There are enough lines in order that each polynomial is algebraically essentially unique. Moreover every polynomial topologically equivalent to f_s is algebraic equivalent to a $f_{s'}$, where s' may be different from s.

For generic parameters the polynomials are topologically equivalent all together and the function f_s is a Morse function on $\mathbb{C}^2 \setminus f_s^{-1}(0)$. We choose our counter-examples with non-generic parameters, for such an example f_k is not a Morse function on $\mathbb{C}^2 \setminus f_k^{-1}(0)$. The fact that non-generic parameters are finite enables us to prove the requested properties.

4. Non reality

Let
$$f_s(x,y) = xy(x-y)(y-1)(x-sy).$$
Let $k, \bar{k}$ be the roots of $s^2 - s + 1$.

Theorem A. *There does not exist a polynomial g with real coefficients such that $g \approx f_k$.*

Let $\mathcal{C} = \{0, 1, k, \bar{k}\}$. Then for $s \in \mathbb{C} \setminus \mathcal{C}$, f_s verifies $\mu = 14$, $\#\mathcal{B}_{\mathit{aff}} = 3$ and $\mathcal{B}_\infty = \varnothing$. By the connectivity of $\mathbb{C} \setminus \mathcal{C}$ and the global version of the μ-constant theorem, two polynomials f_s and $f_{s'}$, with $s, s' \notin \mathcal{C}$, are topologically equivalent.

The polynomials f_k and $f_{\bar{k}}$ verify $\mu = 14$, but $\#\mathcal{B}_{\mathit{aff}} = 2$. Then such a polynomial is not topologically equivalent to a generic one f_s, $s \notin \mathcal{C}$. In fact for $s \notin \mathcal{C}$ there are two non-zero critical fibers with one double point for each one. For $s = k$ or $s = \bar{k}$, there is only one non-zero critical fiber with an ordinary cusp.

Lemma 1. *Let $s, s' \in \mathbb{C}$. The polynomials f_s and $f_{s'}$ are algebraically equivalent if and only if $s = s'$ or $s = 1 - s'$.*

In particular the polynomials f_k and $f_{\bar{k}}$ are algebraically equivalent.

Proof. Let us suppose that f_s and $f_{s'}$ are algebraically equivalent. Then we can suppose that there exists $\Phi \in \operatorname{Aut} \mathbb{C}^2$ such that $f_{s'} = f_s \circ \Phi$. Such a Φ must send the lines $(x = 0), (y = 0)$ to two lines, then Φ is linear: $\Phi(x,y) = (ax + by, cx + dy)$. A calculus proves that $\Phi(x,y) = (x,y)$ or $\Phi(x,y) = (y - x, y)$ that is to say $s = s'$ or $s = 1 - s'$. $\square$

Lemma 2. *Fix $s \in \mathbb{C}$ and let f be a polynomial such that $f \approx f_s$. There exists s' such that $f \sim f_{s'}$.*

Then lemma 1 implies that there are only two choices for s', but s' can be different from s.

Proof. The curve $f_s^{-1}(0)$ contains the simply connected curve $xy(x-y)(x-sy)$, then the curve $f^{-1}(0)$ contains also a simply connected curve (with 4 components), by the generalization of Zaĭdenberg-Lin theorem (see [4]) this simply connected curve is algebraically equivalent to $xy(x-y)(x-s'y)$. Then the polynomial f is algebraically equivalent to $xy(x-y)(x-s'y)P(x,y)$. The curve C defined by $(P = 0)$ is homeomorphic to $\mathbb{C}$ and admits a polynomial parameterization $(\alpha(t), \beta(t))$ with $\alpha, \beta \in \mathbb{C}[t]$. Since C does not intersect the axe $(y = 0)$, β is a constant polynomial; since C intersects the axe $(x = 0)$ at one point α is monomial. An equation of P is now $P(x, y) = y^n - \lambda$. By the irreducibility of C and up to an homothety we get $P(x, y) = y - 1$. That is to say f is algebraically equivalent to $f_{s'}$. □

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Let $g \in \mathbb{C}[x, y]$, if $g(x, y) = \sum a_{i,j} x^i y^j$ then we denote by $\bar{g}$ the polynomial defined by $\bar{g}(x, y) = \sum \bar{a}_{i,j} x^i y^j$. Then $g = \bar{g}$ if and only if all the coefficients of g are real.

We prove theorem A. Let suppose that there exists a polynomial g such that $g = \bar{g}$ and $g \approx f_k$. There exists $s \in \mathbb{C}$ such that $g \sim f_s$. Since f_k has only two critical values, g and f_s have two critical values. Then $s = k$ or $s = \bar{k}$ ($s = 0$ or $s = 1$ gives a polynomial with non-isolated singularities). As $f_k \sim f_{\bar{k}}$ we can choose $s = k$. As a consequence we have $\Phi \in \operatorname{Aut} \mathbb{C}^2$ such that $g = f_k \circ \Phi$.

Let Φ be $\Phi = (p, q)$. Then $g = pq(p - q)(q - 1)(p - kq)$. As $g = \bar{g}$ we have :

$$\big\{p, q, p - q, q - 1, p - kq\big\} = \big\{\bar{p}, \bar{q}, \bar{p} - \bar{q}, \bar{q} - 1, \bar{p} - \bar{k}\bar{q}\big\}.$$

Moreover by the configuration of the lines we have that $q - 1 = \bar{q} - 1$. So $q = \bar{q}$. Hence $q \in \mathbb{R}[x, y]$. So

$$\big\{p, p - q, p - kq\big\} = \big\{\bar{p}, \bar{p} - \bar{q}, \bar{p} - \bar{k}\bar{q}\big\}.$$

Let suppose that $p \neq \bar{p}$. Then $p = \bar{p} - q$ or $p = \bar{p} - \bar{k}q$. So $p - \bar{p}$ equals $-q$ or $-\bar{k}q$. But $p - \bar{p}$ has coefficients in $i\mathbb{R}$, which is not the case of $q \in \mathbb{R}[x, y]$ nor of $\bar{k}q$. Then $p = \bar{p}$. We have proved that $\Phi = (p, q)$ has real coefficients. From $g = f_k \circ \Phi$ we get $\bar{g} = \bar{f_k} \circ \bar{\Phi}$. So $g = f_{\bar{k}} \circ \Phi$. On the one hand $f_k = g \circ \Phi^{-1}$ and on the other hand $f_{\bar{k}} = g \circ \Phi^{-1}$. So $f_k = f_{\bar{k}}$, then $k = \bar{k}$ which is false. It ends the proof.

We could have end in the following way: $\Phi = (p, q)$ is in $\operatorname{Aut} \mathbb{C}^2$ with real coefficients, then Φ, considered as a real map, is in $\operatorname{Aut} \mathbb{R}^2$ (see [3, Theorem 2.1] for example). Then $f_k = g \circ \Phi^{-1}$ with g, Φ^{-1} with real coefficients, then f_k has real coefficients which provides the contradiction.

6. Non connectivity

Let

$$f_s(x, y) = xy(y - 1)(x + y - 1)(x - sy).$$

Let $\mathcal{C}$ be the roots of

$$s(s-1)(s+1)(256s^4+736s^3+825s^2+736s+256)(256s^4+448s^3+789s^2+448s+256).$$

Then for $s \in \mathbb{C} \setminus \mathcal{C}$, f_s verifies $\mu = 14$, $\#\mathcal{B}_{\mathit{aff}} = 4$ and $\mathcal{B}_\infty = \varnothing$. For $s, s' \notin \mathcal{C}$, f_s and $f_{s'}$ are topologically equivalent. The roots of $256s^4 + 448s^3 + 789s^2 + 448s + 256$ are of the form $\{k, \bar{k}, 1/k, 1/\bar{k}\}$. The polynomials f_k and $f_{\bar{k}}$ verify $\mu = 14$, but $\#\mathcal{B}_{\mathit{aff}} = 3$. Then such a polynomial is not topologically equivalent to a generic one f_s, $s \notin \mathcal{C}$

Theorem B. *The polynomials f_k and $f_{\bar{k}}$ are topologically equivalent and it is not possible to find a continuous family $(g_t)_{t\in[0,1]}$ such that $g_0 = f_k$, $g_1 = f_{\bar{k}}$ and $g_t \approx f_k$ for all $t \in [0, 1]$.*

The polynomials f_k and $f_{\bar{k}}$ are topologically equivalent since we have the formula $f_{\bar{k}}(\bar{x}, \bar{y}) = \overline{f_k(x, y)}$.

The two following lemmas are similar to lemmas 1 and 2.

Lemma 3. *The polynomials f_s and $f_{s'}$ are algebraically equivalent if and only if $s = s'$ or $s = 1/s'$.*

Lemma 4. *Fix s and let f be a polynomial such that $f \approx f_s$. Then there exists s' such that $f \sim f_{s'}$.*

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We now prove theorem B. Let us suppose that such a family (g_t) does exist. Then by lemma 4 for each $t \in [0, 1]$ there exists $s(t) \in \mathbb{C}$ such that g_t is algebraically equivalent to $f_{s(t)}$ (in fact there are two choices for $s(t)$). We can suppose that there exists $\Phi_t \in \operatorname{Aut}\mathbb{C}^2$ such that $f_{s(t)} = g_t \circ \Phi_t$.

We now prove that the map $t \mapsto \Phi_t$ can be chosen continuous, that is to say the coefficients of the defining polynomials are continuous functions of t. We write $g_t = A_t B_t G_t$ such that $A_0(x, y) = x$, $B_0(x, y) = y$ and the maps $t \mapsto A_t$, $t \mapsto B_t$ are continuous. So the automorphism Φ_t^{-1} is defined by

$$\Phi_t^{-1}(x, y) = \big(A_t(x, y), B_t(x, y)\big).$$

By the inverse local theorem with parameter t, we have that $t \mapsto \Phi_t$ is a continuous function. Then the map $t \mapsto f_{s(t)}$ is a continuous function, as the composition of two continuous functions. As $s(t)$ is a coefficient of the polynomial $f_{s(t)}$, the map $t \mapsto s(t)$ is a continuous function.

As a conclusion we have a map $t \mapsto s(t)$ which is continuous and such that $s(0) = k$ and $s(1) = \bar{k}$. It implies that there exists $\tau \in]0, 1[$ such that $s(\tau) \notin \mathcal{C}$. On the one hand g_τ is algebraically, hence topologically, equivalent to $f_{s(\tau)}$; on the other hand g_τ is topologically equivalent to f_k (by hypothesis). As $s(\tau) \notin \mathcal{C}$, $f_{s(\tau)}$ and f_k are not topologically equivalent (because $\#\mathcal{B}_{\mathit{aff}}$ are different), it provides a contradiction.

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The calculus have been done with the help of SINGULAR, [8], and especially with author's library `critic` described in [6].

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Topological equivalence of complex polynomials

avec Mihai Tibar

TOPOLOGICAL EQUIVALENCE OF COMPLEX POLYNOMIALS

ARNAUD BODIN AND MIHAI TIBĂR

Abstract. The following numerical control over the topological equivalence is proved: two complex polynomials in $n \neq 3$ variables and with isolated singularities are topologically equivalent if one deforms into the other by a continuous family of polynomial functions $f_s\colon \mathbb{C}^n \to \mathbb{C}$ with isolated singularities such that the degree, the number of vanishing cycles and the number of atypical values are constant in the family.

1. Introduction

Two polynomial functions $f, g : \mathbb{C}^n \to \mathbb{C}$ are said to be *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ and $\Psi : \mathbb{C} \to \mathbb{C}$ such that $\Psi \circ f = g \circ \Phi$. A challenging natural question is: *under what conditions this topological equivalence is controlled by numerical invariants?*

We shall assume that our polynomials have isolated critical points, and therefore finitely many. It appears that the topology of a polynomial function depends not only on critical points but also, since the function is non-proper, on the behaviour of its fibres in the neighbourhood of infinity. It is well-known (and goes back to Thom [21]) that a polynomial function has a finite set of atypical values $\mathcal{B} \subset \mathbb{C}$, i.e. values at which the function fails to be a locally trivial fibration. Even if the critical points of the polynomial are isolated, the homology of fibres may be very complicated, behaving as if highly non-isolated singularities occur at infinity. One has studied such kind of singularities at infinity in case they are in a certain sense isolated, e.g. [4], [19], [16], [20]. In this case the reduced homology of the general fibre G is concentrated in dimension $n-1$ and certain numbers may be attached to singular points at infinity.

Coming back to topological equivalence: if our f and g are topologically equivalent then clearly their corresponding fibres (general or atypical) are homeomorphic. In particular the Euler characteristics of the general fibres of f and g and the numbers of atypical values of f

and g coincide respectively. We prove the following numerical criterion for topological equivalence (see §2 for an example):

Theorem 1. *Let $(f_s)_{s\in[0,1]}$ be a continuous family of complex polynomials with isolated singularities in the affine space and at infinity, in $n \neq 3$ variables. If the numbers $\chi(G(s))$, $\#\mathcal{B}(s)$ and $\deg f_s$ are independent of $s \in [0,1]$, then the polynomials f_0 and f_1 are topologically equivalent.*

In case of a smooth family of germs of holomorphic functions with isolated singularity $g_s : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, a famous result by Lê D.T. and C.P. Ramanujam [13] says that the constancy of the *local Milnor number* (equivalently, of the Euler characteristic of the general fibre in the local Milnor fibration [17]) implies that the hypersurface germs $g_0^{-1}(0)$ and $g_1^{-1}(0)$ have the same topological type whenever $n \neq 3$. J.G. Timourian [23] and H. King [9] showed moreover the topological triviality of the family of function germs over a small enough interval. The techniques which are by now available for proving the Lê-Ramanujam-Timourian-King theorem do not work beyond the case of isolated singularities. In other words, the topological equivalence problem is still unsolved for local non-isolated singularities.

The global setting poses new problems since one has to deal in the same time with several singular points and atypical values. Singularities at infinity introduce a new and essential difficulty since they are of a different type than the critical points of holomorphic germs. Some evidence for the crucial importance of singularities at infinity, even when assumed isolated, in understanding the behaviour of polynomials is the famous unsolved Jacobian Conjecture. One of the equivalent formulations of this conjecture, in two variables, is the following, see [14], [19]: if $f : \mathbb{C}^2 \to \mathbb{C}$ has no critical points but has isolated singularities at infinity then, for any polynomial $h : \mathbb{C}^2 \to \mathbb{C}$, the critical locus $Z(\mathrm{Jac}(f, h))$ is not empty.

Our approach consists in three main steps, which we briefly describe in the following.

Step 1. We show how the assumed invariance of numbers implies the rigidity of the singularities, especially of those at infinity. We use the following specific definition of *isolated singularities at infinity*, also employed in other papers (see e.g. Libgober [15] and Siersma-Tibăr [20]): the projective closure of any fibre of the polynomial and its slice by the hyperplane at infinity have isolated singularities. We explain in §3 how this condition enters in the proof of the key results on the semi-continuity of certain numbers, which in turn implies the rigidity of singularities. The class of polynomials with isolated singularities at infinity is large enough to include all polynomials of two variables with reduced fibres.

Step 2. Assuming the rigidity of some singularity, we prove a local topological triviality result. In case of a singular point in $\mathbb{C}^n$ this is Timourian's theorem, but we have to consider the new case of a singularity at infinity. To handle such a problem we first compactify all the polynomials of the family in the "same way", i.e. we consider the total space $\mathbb{X}$ of a family depending polynomially on the parameter: here we need the constancy of the degree. One cannot conclude by Thom-Mather's first Isotopy Lemma since the natural stratification given by the trajectory of the singular point is not Whitney in general. Unlike the local case, in our setting the underlying space $\mathbb{X}$ turns out to be singular (essentially since the compactification has singularities at infinity). Our strategy is to revisit and modify the explicit local trivialisation given by Timourian's proof by taking into account the Whitney stratification of $\mathbb{X}$ (§5). The use of Timourian's proof is also responsible for the excepted case $n = 3$, due to an argument by Lê-Ramanujam which relies on the h-cobordism theorem, cf [13].

Step 3. Finally, we show how to patch together all the pieces (i.e. some open subsets of $\mathbb{C}^n$) where we have topological triviality, in order to obtain the global topological equivalence. The first named author used patching in [2] to prove topological equivalence in case there are no singularities at infinity and in case $n = 2$ with additional hypotheses and relying on results by L. Fourrier [7] which involve resolution of singularities. In our setting we have to deal with pieces coming from singularities at infinity and their patching is more delicate (see §6).

Let us remark that our theorem only requires the continuity of the family instead of the smoothness in [13], [23]. The reduction from a continuous family to a family depending polynomially on the parameter is made possible by a constructibility argument developed in §4. The constructibility also implies the finiteness of topological types of polynomials when fixing numerical invariants, see Remark 8. It is worth to point out that the finiteness does not hold for the equivalence up to diffeomorphisms, as already remarked by T. Fukuda [8]. For example, the family $f_s(x, y) = xy(x - y)(x - sy)$ provides infinitely many classes for this equivalence, because of the variation of the cross-ratio of the 4 lines.

2. Definitions and notations

We consider a one-parameter family of polynomials $f_s(x) = P(x,s)$, where $P : \mathbb{C}^n \times [0,1] \to \mathbb{C}$ is polynomial in s and such that $\deg f_s = d$, for all $s \in [0,1]$.

We assume that the *affine singularities of f_s are isolated*: $\dim \operatorname{Sing} f_s \leqslant 0$ for all s, where $\operatorname{Sing} f_s = \{x \in \mathbb{C}^n \mid \operatorname{grad} f_s(x) = 0\}$. The set of *affine critical values* of f_s is a finite set and we denote it by $\mathcal{B}_{\mathit{aff}}(s) = \{t \in \mathbb{C} \mid \mu_t(s) > 0\}$, where $\mu_t(s)$ is the sum of the local Milnor numbers at the singular points of the fibre $f_s^{-1}(t)$; remark that we also have $\mathcal{B}_{\mathit{aff}}(s) = f_s(\operatorname{Sing} f_s)$. The *total Milnor number* is $\mu(s) = \sum_{t \in \mathcal{B}_{\mathit{aff}}} \mu_t(s)$. We also assume that, for all s, f_s has isolated singularities at infinity in the following sense.

Definition 2. We say that a polynomial f_s has *isolated singularities at infinity* if $\dim \operatorname{Sing} W(s) \leqslant 0$, where

$$W(s) = \left\{ [x] \in \mathbb{P}^{n-1} \mid \frac{\partial P_d}{\partial x_1} = \cdots = \frac{\partial P_d}{\partial x_n} = 0 \right\}$$

is an algebraic subset of the hyperplane at infinity H^∞ of $\mathbb{P}^n$, which we identify to $\mathbb{P}^{n-1}$. Here P_d denotes the homogeneous part of degree d in variables $x = (x_1, \ldots, x_n)$ of the polynomial $P(x,s)$. The condition $\operatorname{Sing} W(s) \leqslant 0$ is equivalent to the following: for all $t \in \mathbb{C}$, the singularities of $\overline{f_s^{-1}(t)}$ and of $\overline{f_s^{-1}(t)} \cap H^\infty$ are at most isolated.

In [20] one calls "$\mathcal{F}$ISI deformation of f_0" a family P such that f_s has isolated singularities at infinity and in the affine, for all s. The class of polynomials with isolated singularities at infinity is large enough to include all polynomial functions in two variables with reduced fibres. It is a (strict) subclass of polynomials having isolated singularities at infinity in the sense used by Broughton [4] or in the more general sense of [19].

We shall see in the following how one can precisely detect the singularities at infinity. We attach to the family P the following hypersurface:

$$\mathbb{X} = \left\{ ([x : x_0], t, s) \in \mathbb{P}^n \times \mathbb{C} \times [0,1] \mid \tilde{P}(x, x_0, s) - t x_0^d = 0 \right\},$$

where $\tilde{P}$ denotes the homogenisation of P by the new variable x_0, considering s as parameter. Let $\tau : \mathbb{X} \to \mathbb{C}$ be the projection to the t-coordinate. This extends the map P to a proper one in the sense that $\mathbb{C}^n \times [0,1]$ is embedded into $\mathbb{X}$ and that the restriction of τ to $\mathbb{C}^n \times [0,1]$ is equal to P. Let $\sigma : \mathbb{X} \to [0,1]$ denote the projection to the s-coordinate. We shall use the notations $\mathbb{X}(s) = \sigma^{-1}(s) \cap \mathbb{X}$.

Let $\mathbb{X}_t(s) = \mathbb{X}(s) \cap \tau^{-1}(t)$ be the projective closure in $\mathbb{P}^n$ of the affine hypersurface $f_s^{-1}(t)$. Note that $\mathbb{X}(s)$ is singular, in general with 1-dimensional singular locus, since $\operatorname{Sing}\mathbb{X}(s) = \Sigma(s) \times \mathbb{C}$, where:

$$\Sigma(s) = \left\{ [x] \in \mathbb{P}^{n-1} \mid \frac{\partial P_d}{\partial x_1}(x,s) = \cdots = \frac{\partial P_d}{\partial x_n}(x,s) = P_{d-1}(x,s) = 0 \right\}$$

and we have $\Sigma(s) \subset W(s)$, which implies that $\Sigma(s)$ is finite.

Let us fix some $s \in [0,1]$ and some $p \in \Sigma(s)$. For $t \in \mathbb{C}$, let $\mu_p(\mathbb{X}_t(s))$ denote the local Milnor number of the projective hypersurface $\mathbb{X}_t(s) \subset \mathbb{P}^n$ at the point $[p : 0]$. By [4] the number $\mu_p(\mathbb{X}_t(s))$ is constant for generic t, and we denote this value by $\mu_{p,gen}(s)$. We have $\mu_p(\mathbb{X}_t(s)) > \mu_{p,gen}(s)$ for a finite number of values of t. The *Milnor-Lê number* at the point $([p:0],t) \in \mathbb{X}(s)$ is defined as the jump $\lambda_{p,t}(s) := \mu_p(\mathbb{X}_t(s)) - \mu_{p,gen}(s)$. We say that the point $([p:0],t)$ is a *singularity at infinity* of f_s if $\lambda_{p,t}(s) > 0$. Let $\lambda_t(s) = \sum_{p\in\Sigma(s)} \lambda_{p,t}(s)$. The set of *critical values at infinity* of the polynomial f_s is defined as:

$$\mathcal{B}_{inf}(s) = \{t \in \mathbb{C} \mid \lambda_t(s) > 0\}.$$

Finally, the *Milnor-Lê number at infinity* of f_s is defined as:

$$\lambda(s) = \sum_{t\in\mathcal{B}_{inf}(s)} \lambda_t(s).$$

For such a polynomial, the *set of atypical values*, or the *bifurcation set*, is:

$$\mathcal{B}(s) = \mathcal{B}_{aff}(s) \cup \mathcal{B}_{inf}(s).$$

It is known that $f_s : f_s^{-1}(\mathbb{C}\backslash\mathcal{B}(s)) \to \mathbb{C}\backslash\mathcal{B}(s)$ is a locally trivial fibration [4]. After [19], for $t \in \mathbb{C}$ the fibre $f_s^{-1}(t)$ is homotopic to a wedge of spheres of real dimension $n-1$ and the number of these spheres is $\mu(s) + \lambda(s) - \mu_t(s) - \lambda_t(s)$. In particular, for the Euler characteristic of the general fibre $G(s)$ of f_s one has:

$$\chi(G(s)) = 1 + (-1)^{n-1}(\mu(s) + \lambda(s)).$$

Example 3. Let $f_s(x,y,z,w) = x^2y^2 + z^2 + w^2 + xy + (1+s)x^2 + x$. For $s \in \mathbb{C} \setminus \{-2,-1\}$ we have $\mathcal{B}(s) = \left\{0, -\frac{1}{4}, -\frac{1}{4}\frac{s+2}{s+1}\right\}$, $\mu(s) = 2$ and $\lambda(s) = 1$. It follows that $\chi(G(s)) = 1 - \mu(s) - \lambda(s) = -2$ and that $\#\mathcal{B}(s) = 3$. For the two excepted polynomials f_{-1} and f_{-2} we have $\#\mathcal{B} = 2$. Then, by Theorem 1, f_0 is topologically equivalent to f_s if and only if $s \in \mathbb{C} \setminus \{-2,-1\}$.

Let $\mathbb{X}^\infty(s)$ denote the part "at infinity" $\mathbb{X}(s) \cap \{x_0 = 0\}$. We shall use in §5 the Whitney stratification of the space $\mathbb{X}(s)$ with the following strata (see [19], [20]): $\mathbb{X}(s) \setminus \mathbb{X}^\infty(s)$, $\mathbb{X}^\infty(s) \setminus \operatorname{Sing}\mathbb{X}(s)$, the complement in $\operatorname{Sing}\mathbb{X}(s)$ of the singularities at infinity and the finite set of singular points at infinity. We also recall that the restriction $\tau : \mathbb{X}(s) \to \mathbb{C}$ is transversal to all the strata of $\mathbb{X}(s)$ except at the singular points at infinity.

3. Rigidity of singularities in families of polynomials

Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials with constant degree d, such that the coefficients of f_s are polynomial functions of $s \in [0,1]$. We also suppose that for all $s \in [0,1]$, f_s has isolated singularities in the affine space and at infinity (in the sense of Definition 2). Under these conditions, we may prove the following rigidity result:

Proposition 4. *If the pair of numbers $(\mu(s)+\lambda(s), \#\mathcal{B}(s))$ is independent of s in some neighbourhood of 0, then the following five-uple $(\mu(s), \#\mathcal{B}_{\mathit{aff}}(s), \lambda(s), \#\mathcal{B}_{\mathit{inf}}(s), \#\mathcal{B}(s))$ is independent of s too. Moreover there is no collision of points $p(s) \in \Sigma(s)$ as $s \to 0$, and in particular $\#\Sigma(s)$ and $\mu_{p,gen}(s)$ are constant.*

Proof. **Step 1.** We claim that the multivalued map $s \mapsto \mathcal{B}(s)$ is continuous. If not the case, then there is some value of $\mathcal{B}(s)$ which disappears as $s \to 0$. To compensate this, since $\#\mathcal{B}(s)$ is constant, there must be a value which appears in $\mathcal{B}(0)$. By the local constancy of the total Minor number, affine singularities cannot appear from nothing, therefore the new critical value should be in $\mathcal{B}_{\mathit{inf}}(0)$. More precisely, there is a singular point at infinity (p,t) of f_0 (thus where the local λ is positive) such that, for $s \neq 0$, there is no singular point of f_s, either in affine space or at infinity, which tends to (p,t) as $s \to 0$. But this situation has been shown to be impossible in [2, Lemma 20]. Briefly, the argument goes as follows: Let $(p, c(0))$ be a singularity at infinity of f_0 and let $h_{s,t} : \mathbb{C}^n \to \mathbb{C}$ be the localisation at p of the map $\tilde{P}(x, x_0, s) - tx_0^d$. Then from the local conservation of the total Milnor number of $h_{0,c(0)}$ and the dimension of the critical locus of the family $h_{s,t}$ one draws a contradiction. The claim is proved.

Let us remark that our proof also implies that the finite set $\mathcal{B}(s) \subset \mathbb{C}$ is contained in some disk of radius independent of s and that there is no collision of points of $\mathcal{B}(s)$ as $s \to 0$.

Step 2. We prove that there is no collision of points $p(s) \in \Sigma(s)$ as $s \to 0$ and that $\#\Sigma(s)$ and $\mu_{p,gen}(s)$ are constant. We pick up and fix a value $t \in \mathbb{C}$ such that $t \notin \mathcal{B}(s)$, for all s near 0. Then we have a one parameter family of general fibres $f_s^{-1}(t)$, where s varies in a neighbourhood of 0. The corresponding compactified hypersurfaces $\mathbb{X}_t(s)$ have isolated singularities at their intersections with the hyperplane at infinity H^∞.

Let $\mu_p^\infty(s)$ denote the Milnor number of the hyperplane slice $\mathbb{X}_t(s) \cap H^\infty$ at some $p \in W(s)$, and note that this does not depend on t, for some fixed s. We use the following formula (see [20, 2.4] for the proof and references):

$$\mu(s) + \lambda(s) = (d-1)^n - \sum_{p \in \Sigma(s)} \mu_{p,gen}(s) - \sum_{p \in W(s)} \mu_p^\infty(s). \tag{3.1}$$

Since $\mu(s)+\lambda(s)$ is constant and since the local upper semi-continuity of Milnor numbers, we have that both sums $\sum_{p\in\Sigma(s)} \mu_{p,gen}(s)$ and $\sum_{p\in W(s)} \mu_p^\infty(s)$ are constant hence locally constant. The non-splitting principle (see [10] or [12], [1]) applied to our family of hypersurface multigerms tells that each $\mu_{p,gen}(s)$ has to be constant. This means that there cannot be collision of points of $\Sigma(s)$.

Step 3. We claim that $\mu(s)$ is constant. If not the case, then we may suppose that $\mu(0) < \mu(s)$, for s close to 0, since $\mu(s)$ is lower semi-continuous (see [4]). Then by also using Step 1, there exists $c(s) \in \mathcal{B}_{\mathit{aff}}(s)$, such that: $c(s) \to c(0) \in \mathbb{C}$ as $s \to 0$. By Step 1, there is no other value except $c(s) \in \mathcal{B}(s)$ which tends to $c(0)$. We therefore have a family of hypersurfaces $\mathbb{X}_{c(s)}(s)$ with isolated singularities $q_j(s) \in f_s^{-1}(c(s))$ that tend to the singularity at $(p, 0) \in \Sigma(0) \subset \mathbb{X}_{c(0)}(0)$. By Step 2 and the (upper) semi-continuity of the local Milnor numbers we have:

$$\mu_p(\mathbb{X}_{c(0)}(0)) \geqslant \mu_p(\mathbb{X}_{c(s)}(s)) + \sum_j \mu_{q_j(s)}(\mathbb{X}_{c(s)}(s)). \tag{3.2}$$

By definition, $\mu_p(\mathbb{X}_{c(s)}(s)) = \lambda_p(s) + \mu_{p,gen}(s)$ for any s, and by Step 2, $\mu_{p,gen}(s)$ is independent of s. It follows that:

$$\lambda_{p,c(0)}(0) \geqslant \lambda_{p,c(s)}(s) + \sum_j \mu_{q_j(s)}(\mathbb{X}_{c(s)}(s)), \tag{3.3}$$

which actually expresses the balance at any collision of singularities at some point at infinity. This shows that in such collisions the "total quantity of singularity", i.e. the local $\mu + \lambda$, is upper semi-continuous. On the other hand, the global $\mu + \lambda$ is assumed constant, by our hypothesis. This implies that the local $\mu + \lambda$ is constant too. Therefore in (3.3) we must have equality and consequently (3.2) is an equality too.

We may now conclude by applying the non-splitting principle, similarly as in Step 2, to yield a contradiction.

Step 4. Since by Step 3 there is no loss of μ, the multi-valued function $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is continuous. Steps 1 and 3 show that $s \mapsto \mathcal{B}_{\mathit{inf}}(s)$ is continuous too. Together with $\#(\mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_{\mathit{inf}}(s)) = \text{cst}$, this implies that $\#\mathcal{B}_{\mathit{aff}}(s) = \text{cst}$ and $\#\mathcal{B}_{\mathit{inf}}(s) = \text{cst}$. □

4. Constructibility via numerical invariants

Let $\mathcal{P}_{\leqslant d}$ be the vector space of all polynomials in n complex variables of degree at most d. We consider here the subset $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B}) \subset \mathcal{P}_{\leqslant d}$ of polynomials of degree d with fixed $\mu + \lambda$ and fixed $\#\mathcal{B}$.

Recall that a *locally closed set* is the intersection of a Zariski closed set with a Zariski open set; a *constructible set* is a finite union of locally closed sets.

Proposition 5. *$\mathcal{P}_d(\mu + \lambda, \#\mathcal{B})$ is a constructible set.*

Proof. The set $\mathcal{P}_d$ of polynomials of degree d is a constructible set in the vector space $\mathcal{P}_{\leqslant d}$. Let us first prove that "isolated singularities at infinity" yields a constructible set. A polynomial f has isolated singularities at infinity if and only if $W := W(f)$ has dimension 0 or is void. Let $S = \{(x, f) \in \mathbb{P}^n \times \mathcal{P}_d \mid f \in \mathcal{P}_d,\ x \in W(f)\}$ and let $\pi : S \to \mathcal{P}_d$ be the projection on the second factor. Since this is an algebraic map, by Chevalley's Theorem (e.g. [6, §14.3]) the set $\{f \in \mathcal{P}_d \mid \dim \pi^{-1}(f) \leqslant 0\}$ is constructible and this is exactly the set of polynomials with isolated singularities at infinity.

Next, we prove that fixing each integer μ, $\#\mathcal{B}_{\mathit{aff}}$, λ, $\#\mathcal{B}_{\mathit{inf}}$, $\#\mathcal{B}$ yields a constructible set. The main reason is the semi-continuity of the Milnor number (upper in the local case, lower in the affine case), see e.g. Broughton [4, Prop. 2.3]. Broughton proved that the set of polynomials with a given $\mu < \infty$ is constructible. As the inverse image of a constructible set by an algebraic map, the set of polynomials with Milnor number μ and bifurcation set such that $\#\mathcal{B}_{\mathit{aff}} = k$ is a constructible set.

Let $\mathcal{P}_d(\mu, \#\Sigma)$ be the set of polynomials of degree d, with a given μ, with isolated singularities at infinity and a given $\#\Sigma$. Notice that $\#\Sigma$ is finite because $\Sigma \subset W$ and is bounded for fixed d. Since Σ depends algebraically on f, we have that $\mathcal{P}_d(\mu, \#\Sigma)$ is a constructible set. Now the local Milnor number μ_p is an upper semi-continuous function, so fixing λ_p as the difference of two local Milnor numbers (see §2) provides a constructible set. By doing this for all the critical points at infinity we get that fixing $\lambda = \sum_p \lambda_p$ yields a constructible condition. The arguments for the conditions $\#\mathcal{B}_{\mathit{inf}}$ and $\#\mathcal{B}$ (which are numbers of points of two algebraic sets in $\mathbb{C}$) are similar to the one for $\#\mathcal{B}_{\mathit{aff}}$.

The just proved constructibility of $\mathcal{P}_d(\mu, \#\mathcal{B}_{\mathit{aff}}, \lambda, \#\mathcal{B}_{\mathit{inf}}, \#\mathcal{B})$ implies, by taking a finite union, the constructibility of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$. □

Definition 6. We say that a finite set $\Omega(s)$ of points in $\mathbb{C}^k$, for some k, depending on a real parameter s, is an *algebraic braid* if $\Omega = \cup_s \Omega(s) \times \{s\}$ is a real algebraic sub-variety of $\mathbb{C}^k \times [0,1]$, the multi-valued function $s \mapsto \Omega(s)$ is continuous and $\#\Omega(s) = \mathrm{cst}$.

We may now reformulate and extend Proposition 4 as follows.

Proposition 7. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials with isolated singularities in the affine space and at infinity, whose coefficients are polynomial functions of s. Suppose that the numbers $\mu(s) + \lambda(s)$, $\#\mathcal{B}(s)$ and $\deg f_s$ are independent of $s \in [0,1]$. Then:*

(1) *$\Sigma(s)$, $\mathcal{B}_{\mathit{aff}}(s)$, $\mathcal{B}_{\mathit{inf}}(s)$ and $\mathcal{B}(s)$ are algebraic braids;*
(2) *for any continuous function $s \mapsto p(s) \in \Sigma(s)$ we have $\mu_{p(s),gen} = \mathrm{cst}$;*
(3) *for any continuous function $s \mapsto c(s) \in \mathcal{B}_{\mathit{inf}}(s)$ we have $\lambda_{p(s),c(s)} = \mathrm{cst}$;*
(4) *for any continuous function $s \mapsto c(s) \in \mathcal{B}_{\mathit{aff}}(s)$ we have $\mu_{c(s)} = \mathrm{cst}$ and moreover, the local μ's of the fibre $f_s^{-1}(c(s))$ are constant.*

Proof. (1) For $\Sigma(s)$, it follows from the algebraicity of the definition of Σ and from Step 2 of Proposition 4. It is well-known that affine critical values of polynomials are algebraic functions of the coefficients. Together with Proposition 4, this proves that $\mathcal{B}_{\mathit{aff}}(s)$ is an algebraic braid.

Similarly $\cup_s \mathcal{B}_{\mathit{inf}}(s) \times \{s\}$ is the image by a finite map of an algebraic set, and together with Step 4 of Proposition 4, this proves that $\mathcal{B}_{\mathit{inf}}(s)$ is an algebraic braid.

Next, (2) is Step 2 of Proposition 4 and (3) is a consequence of Step 3. Lastly, observe that (4) is a well-known property of local isolated hypersurface singularities and follows from (1) and the local non-splitting principle. □

Remark 8. Theorem 1 has the following interpretation: to a connected component of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ one associates a unique topological type. (It should be noticed that two different connected components of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ may have the same topological type, see [3] for an example.) It follows that there is a finite number of topological types of complex polynomials of fixed degree and with isolated singularities in the affine space and at infinity. This may be related to the finiteness of topological equivalence classes in $\mathcal{P}_{\leqslant d}$, conjectured by René Thom and proved by T. Fukuda [8].

5. Local triviality at infinity

The aim of this section is to prove a topological triviality statement for a singularity at infinity. Our situation is new since it concerns a family of couples space-function varying with the parameter s and where the space is singular. The proof actually relies on Timourian's proof [23] for germs of holomorphic functions on $\mathbb{C}^n$. Therefore we shall point out where and how this proof needs to be modified, since we have to plug-in a singular stratified space (i.e. the germ of $\mathbb{X}(s)$ at a singularity at infinity) instead of the germ $(\mathbb{C}^n, 0)$ in Timourian's proof.

As before, let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials of degree d and let (p, c) be a singularity at infinity of f_0. Let $g_s : \mathbb{X}(s) \to \mathbb{C}$ be the localisation at $(p(s), c(s))$ of the map $\tau_{|\mathbb{X}(s)}$. We denote by $B_\varepsilon \subset \mathbb{C}^n \times \mathbb{C}$ the closed $2n+2$-ball of radius ε centred at (p, c), such that $B_\varepsilon \cap \mathbb{X}(0)$ is a Milnor ball for g_0. We choose $0 < \eta \ll \varepsilon$ such that we get a Milnor tube $T_0 = B_\varepsilon \cap \mathbb{X}(0) \cap g_0^{-1}(D_\eta(c))$. Then, for all $t \in D_\eta(c)$, $g_0^{-1}(t)$ intersects transversally $S_\varepsilon = \partial B_\varepsilon$. We recall from [19] that $g_0 \colon T_0 \setminus g_0^{-1}(c) \to D_\eta(c) \setminus \{c\}$ is a locally trivial fibration whenever $\lambda_{p,c}(0) > 0$ and $g_0 \colon T_0 \to D_\eta(c)$ is a trivial fibration whenever $\lambda_{p,c}(0) = 0$.

According to Proposition 7(1), by an analytic change of coordinates, we may assume that $(p(s), c(s)) = (p, c)$ for all $s \in [0, u]$, for some small enough $u > 0$. We set $T_s = B_\varepsilon \cap \mathbb{X}(s) \cap g_s^{-1}(D_\eta(c))$ and notice that B_ε does not necessarily define a Milnor ball for g_s whenever $s \neq 0$. For some $u > 0$, let $T = \bigcup_{s\in[0,u]} T_s \times \{s\}$, and let $G : T \to \mathbb{C} \times [0, u]$ be defined by $G(z, s) = (g_s(z), s)$.

The homeomorphisms between the tubes that we consider here are all *stratified*, sending strata to corresponding strata. The stratification of some tube T_s has by definition three strata: $\{T_s \setminus (\{p\} \times D_\eta(c)), \{p\} \times D_\eta(c) \setminus (p, c), (p, c)\}$.

Theorem 9. *Let $f_s(x) = P(x, s)$ be a one-parameter polynomial family of polynomial functions of constant degree, such that the numbers $\mu(s)+\lambda(s)$ and $\#\mathcal{B}(s)$ are independent of s. If $n \neq 3$, then there exists $u > 0$ and a homeomorphism α such that the following diagram commutes:*

$$\begin{array}{ccc} T & \xrightarrow{\ \alpha\ } & T_0 \times [0, u] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g_0 \times \mathrm{id}} \\ D_\eta(c) \times [0, u] & \xrightarrow[\mathrm{id}]{} & D_\eta(c) \times [0, u], \end{array}$$

and such that α sends the strata of every T_s to the corresponding strata of T_0.

Proof. Our point $(p, c) \in \Sigma(0) \times \mathbb{C}$ is such that $\lambda_{p,c}(0) > 0$. We cannot apply directly Timourian's result for the family g_s because each function g_s is defined on a *singular* space germ $(\mathbb{X}(s), (p, c))$, but we can adapt it to our situation. To do this we recall the main lines of this proof and show how to take into account the singularities via the stratification of T_s.

Remark first that (p, c) is the only singularity of g_s in T_s, by the rigidity result Proposition 7. We use the notion of ε-homeomorphism, meaning a homeomorphism which moves every point within a distance no more than $\varepsilon > 0$.

Theorem 9 will follow from Lemma 10 below, once we have proved that the assumptions of this lemma are fulfilled in our new setting. This is a simplified statement of Timourian's Lemma 3 in [23], its proof is purely topological and needs no change.

Lemma 10. ([23, Lemma 3]) *Assume that:*

(1) *The space of stratified homeomorphisms of T_0 into itself, preserving the fibres of g_0, is locally contractible.*
(2) *For any $\varepsilon > 0$ there exists $u > 0$ small enough such that for any $s, s' \in [0, u]$ there is a stratified ε-homeomorphism $h : T_s \to T_{s'}$ with $g_s = g_{s'} \circ h$.*

Then there exists a homeomorphism α as in Theorem 9.

The assumptions (1) and (2) correspond, respectively, to Lemma 1 and Lemma 2 of Timourian's paper [23]. The remaining proof is therefore devoted to showing why the assumptions (1) and (2) are true in our setting.

Condition (1) can be proved as follows. It is well-known that analytic sets have local conical structure [5]. Notice that the stratification $\{T_0 \setminus (\{p\} \times D_\eta(c)), \{p\} \times D_\eta(c) \setminus (p, c), (p, c)\}$ of T_0 is a Whitney stratification (but that this is not necessarily true for tubes T_s with $s \neq 0$). Timourian shows how to construct a vector field on T_0 such that all integral curves end at the central point (p, c). Moreover, this vector field can be chosen such that to respect the Whitney strata. This is the only new requirement that we need to plug in. The rest of Timourian's argument remains unchanged once we have got the vector field, and we give only its main lines in the following. This vector field is used to define a continuous family h_t of homeomorphisms such that $g_0 = g_0 \circ h_t$, which deforms a homeomorphism $h_1 = h$ of T_0 which is the identity at the boundary ∂T_0, to a homeomorphism h_0 which is the identity within a neighbourhood of $(\partial B_\varepsilon \cap T_0) \cup g_0^{-1}(c) \setminus (p, c)$. Next, by using the contracting vector field, one constructs an isotopy of h_0 to the identity, preserving the fibres of g_0. To complete the proof, Timourian shows how to get rid of the auxiliary condition "to be the identity at the boundary ∂T_0" imposed to h, by using Siebenmann's results [18].

Condition (2) now. It will be sufficient to construct homeomorphisms as in (2) from T_0 to T_s for every $s \in [0, u]$, and take u sufficiently small with respect to ε. First remark that for a sufficiently small u, the fibre $g_s^{-1}(t)$ intersects transversally the sphere $S_\varepsilon = \partial B_\varepsilon$, for all $s \in [0, u]$, and for all $t \in D_\eta(c)$. Consequently one may define a homeomorphism $h' : \partial B_\varepsilon \cap T_0 \to \partial B_\varepsilon \cap T_s$. The problem is to extend it to an homeomorphism from T_0 to T_s.

Take a Milnor ball $B_{\varepsilon'} \subset B_\varepsilon$ for g_s at (p, c). It appears that $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(c)$ is diffeomorphic to $(\partial B_\varepsilon \cap g_s^{-1}(c)) \times [0, 1]$. This would be a consequence of the h-cobordism theorem (the condition $n \neq 3$ is needed here) provided that it can be applied. The argument is given by Lê-Ramanujam's in [13] and we show how this adapts to our setting. One first notices that Lê-Ramanujam's argument works as soon as one has the following conditions: for $b \in D_\eta(c)$ and $b \neq c$, the fibres $B_\varepsilon \cap g_0^{-1}(b)$ and $B_{\varepsilon'} \cap g_s^{-1}(b)$ are singular only at (p, b), they are homotopy equivalent to a bouquet of spheres S^{n-1} and the number of spheres is the same. In [13] the fibres are non-singular, but non-singularity is only needed at the intersection with spheres ∂B_ε and $\partial B_{\varepsilon'}$. In our setting both fibres are singular Milnor fibres of functions with isolated singularity on stratified hypersurfaces and in such a case, Lê's result [11] tells that they are, homotopically, wedges of spheres of dimension $n - 1$. Now by [19, Theorem 3.1, Cor. 3.5], the number of spheres is equal to $\lambda_{p,c}(0)$ and $\lambda_{p,c}(s)$ respectively. Since $\lambda_{p,c}(s)$ is independent of s by Proposition 7, these two numbers coincide.

This shows that one may apply the h-cobordism theorem and conclude that there exists a C^∞ function without critical points on the manifold $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(c)$, having as levels $\partial B_\varepsilon \cap g_s^{-1}(c)$ and $\partial B_{\varepsilon'} \cap g_s^{-1}(c)$. This function can be extended, with the same property, first on a thin tube $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(\Delta)$, where Δ is a small enough disk centred at c, then further gluing to the distance function on $B_{\varepsilon'} \cap g_s^{-1}(\Delta)$. This extension plays now the role of the distance function in the construction of the contracting vector field on T_s. Finally, this vector field is used to extend the homeomorphism h from the boundary to the interior of T_s, by a similar construction as the one used in proving condition (1).

The conditions (1) and (2) are now proved and therefore Lemma 10 can be applied. Its conclusion is just Theorem 9. □

6. Proof of the main theorem

We first prove Theorem 1 in case the coefficients of the family P are polynomials in the variable s. The general case of continuous coefficients will follow by a constructibility argument.

6.1. **Transversality in the neighbourhood of infinity.** Let $R_1 > 0$ such that for all $R \geqslant R_1$ and all $c \in \mathcal{B}_{inf}(0)$ the intersection $f_0^{-1}(c) \cap S_R$ is transversal. We choose $0 < \eta \ll 1$ such that for all $c \in \mathcal{B}_{inf}(0)$ and all $t \in D_\eta(c)$ the intersection $f_0^{-1}(t) \cap S_{R_1}$ is transversal. We set

$$K(0) = D \setminus \bigcup_{c \in \mathcal{B}_{inf}(0)} \mathring{D}_\eta(c)$$

for a sufficiently large disk D of $\mathbb{C}$. There exists $R_2 \geqslant R_1$ such that for all $t \in K(0)$ and all $R \geqslant R_2$ the intersection $f_0^{-1}(t) \cap S_R$ is transversal (see [22, Prop. 2.11, Cor. 2.12] for a more general result, or see [2, Lemma 5]).

By Proposition 7, $\mathcal{B}(s)$ is an algebraic braid so we may assume that for a large enough D, $\mathcal{B}(s) \subset \mathring{D}$ for all $s \in [0,u]$. Moreover there exists a diffeomorphism $\chi : \mathbb{C} \times [0,u] \to \mathbb{C} \times [0,u]$ with $\chi(x,s) = (\chi_s(x), s)$ and such that $\chi_0 = \mathrm{id}$, that $\chi_s(\mathcal{B}(s)) = \mathcal{B}(0)$ and that χ_s is the identity on $\mathbb{C} \setminus \mathring{D}$, for all $s \in [0,u]$. We set $K(s) = \chi_s^{-1}(K(0))$.

We may choose u sufficiently small such that for all $s \in [0,u]$, for all $c \in \mathcal{B}_{inf}(0)$ and all $t \in \chi_s^{-1}(D_\eta(c))$ the intersection $f_s^{-1}(t) \cap S_{R_1}$ is transversal. We may also suppose that for all $s \in [0,u]$, for all $t \in K(s)$ the intersection $f_s^{-1}(t) \cap S_{R_2}$ is transversal. Notice that the intersection $f_u^{-1}(t) \cap S_R$ may not be transversal for all $R \geqslant R_2$ and $t \in K(s)$.

6.2. **Affine part.** We denote

$$B'(s) = \big(f_s^{-1}(D) \cap B_{R_1}\big) \cup \big(f_s^{-1}(K(s)) \cap B_{R_2}\big), \quad s \in [0,u].$$

By using Timourian's theorem at the affine singularities and by gluing the pieces with vector fields as done in [2, Lemma 15], we get the following trivialisation:

$$\begin{array}{ccc} B' & \xrightarrow{\ \Omega^{\mathit{aff}}\ } & B'(0) \times [0,u] \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle f_0 \times \mathrm{id}} \\ D \times [0,u] & \xrightarrow[\ \chi\ ]{} & D \times [0,u], \end{array}$$

where $B' = \bigcup_{s \in [0,u]} B'(s) \times \{s\}$ and $F(x,s) = (f_s(x), s)$.

6.3. **At infinity, around an atypical value.** It remains to deal with the part at infinity $f_s^{-1}(D) \setminus \mathring{B}'(s)$ according to the decomposition of D as the union of $K(s)$ and of the disks around each $c \in \mathcal{B}_{inf}(s)$. For each singular point $(p, c(0))$ at infinity we have a Milnor tube $T_{p,0}$ defined by a Milnor ball of radius $\varepsilon(p, c(0))$ and a disk of radius η, small enough in order to be a common value for all such points.

Let g_s be the restriction to $\mathbb{X}(s)$ of the compactification τ of the f_s, let $G : \mathbb{X} \longrightarrow \mathbb{C} \times [0, u]$ be defined by $G(x, s) = (g_s(x), s)$ and let $C'(s) = g_s^{-1}(\chi_s^{-1}(D_\eta(c(0)))) \setminus (\mathring{B}_{R_1} \cup \bigcup_{p(s)} \mathring{T}_{p(s)})$. Now g_s is transversal to the following manifolds: to $T_{p(s)} \cap \partial B_\varepsilon$, for all $s \in [0, u]$, by the definition of a Milnor tube, and to $S_{R_1} \cap C'(s)$, by the definition of R_1. We shall call the union of these sub-spaces the *boundary of* $C'(s)$, denoted by $\delta C'(s)$. Let us recall from §2 the definition of the Whitney stratification on $\mathbb{X}(s)$ and remark that $C'(s) \cap \operatorname{Sing} \mathbb{X}(s) = \emptyset$. Therefore g_s is transversal to the stratum $C'(s) \cap \mathbb{X}^\infty(s)$.

Let $C' = \bigcup_{s \in [0,u]} C'(s) \times \{s\}$ and remark that $C' \cap \operatorname{Sing} \mathbb{X} = \emptyset$ and that the stratification $\{C' \setminus \mathbb{X}^\infty, \mathbb{X}^\infty\}$ is Whitney. Then by our assumptions and for small enough u, the function G has maximal rank on $\bigcup_{s \in [0,u]} \mathring{C}'(s) \times \{s\}$, on its boundary $\delta C' = \bigcup_{s \in [0,u]} \delta C'(s) \times \{s\}$ and on $C' \cap \mathbb{X}^\infty(s)$. By Thom-Mather's first Isotopy Theorem, G is a trivial fibration on $C' \setminus \mathbb{X}^\infty$. More precisely, one may construct a vector field on C' which lifts the vector field $(\frac{\partial \chi_s}{\partial s}, 1)$ of $D \times [0, u]$ and which is tangent to the boundary $\delta C'$ and to $C' \cap \mathbb{X}^\infty$. We may in addition impose the condition that it is tangent to the sub-variety $g_s^{-1}(\partial \chi_s^{-1}(D_\eta(c(0)))) \cap S_{R_2}$. We finally get a trivialisation of C', respecting fibres and compatible with χ.

6.4. **Gluing trivialisations by vector fields.** Since this vector field is constructed such that to coincide at the common boundaries with the vector field defined on each tube T in the proof of Theorem 9, and with the vector field on B' as defined above, this enables one to glue all the resulting trivialisations over $[0, u]$. Namely, for

$$B''(s) := \big(f_s^{-1}(D) \cap B_{R_2}\big) \cup \big(f_s^{-1}(D \setminus \mathring{K}(s))\big) \text{ and } B'' := \bigcup_{s \in [0,u]} B''(s) \times \{s\}$$

we get a trivialisation:

$$\begin{array}{ccc} B'' & \xrightarrow{\ \Omega\ } & B''(0) \times [0, u] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0 \times \mathrm{id}} \\ D \times [0, u] & \xrightarrow[\ \chi\ ]{} & D \times [0, u]. \end{array}$$

This diagram proves the topological equivalence of the maps $f_0 : B''(0) \longrightarrow D$ and $f_u : B''(u) \longrightarrow D$.

6.5. **Extending topological equivalences.** By the transversality of $f_0^{-1}(K(0))$ to the sphere S_R, for all $R \geqslant R_2$, it follows that the map $f_0 : B''(0) \longrightarrow \mathring{D}$ is topologically equivalent to $f_0 : f_0^{-1}(\mathring{D}) \longrightarrow \mathring{D}$, which in turn is topologically equivalent to $f_0 : \mathbb{C}^n \longrightarrow \mathbb{C}$.

We take back the argument for f_0 in §6.1 and apply it to f_u: there exists $R_3 \geqslant R_2$ such that $f_u^{-1}(t)$ intersects transversally S_R, for all $t \in K(u)$ and all $R \geqslant R_3$. Now, with arguments similar to the ones used in the proof of the classical Lê-Ramanujam theorem (see e.g. [22, Theorem 5.2] or [2, Lemma 8] for details), we show that our hypothesis of the constancy of $\mu + \lambda$ allows the application of the h-cobordism theorem on $B'''(u) \setminus \mathring{B}''(u)$, where $B'''(u) = \big(f_s^{-1}(D) \cap B_{R_3}\big) \cup \big(f_s^{-1}(D \setminus \mathring{K}(s))\big)$. Consequently, we get a topological equivalence between $f_u : B''(u) \longrightarrow D$ and $f_u : B'''(u) \longrightarrow D$. Finally $f_u : B'''(u) \longrightarrow \mathring{D}$ is topologically equivalent to $f_u : f_u^{-1}(\mathring{D}) \longrightarrow \mathring{D}$ by the transversality evoked above, and this is in turn topologically equivalent to $f_u : \mathbb{C}^n \longrightarrow \mathbb{C}$.

6.6. **Continuity of the coefficients.** So far we have proved Theorem 1 under the hypothesis that the coefficients of the family P are polynomials in the parameter s. We show in the last part of the proof how to recover the case of continuous coefficients. The following argument was suggested to the first named author by Frank Loray. Let $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ be the set of polynomials of degree d, with isolated singularities in the affine space and at infinity, with fixed number of vanishing cycles $\mu+\lambda$ and with a fixed number of atypical values $\#\mathcal{B}$. Proposition 5 tells that $\mathcal{P}_d(\mu + \lambda, \#\mathcal{B})$ is a constructible set. Since f_0 and f_1 are in the same connected component of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$, we may connect f_0 to f_1 by a family g_s with $g_0 = f_0$ and $g_1 = f_1$ such that the coefficients of g_s are piecewise polynomial functions in the variable s. Using the proof done before for each polynomial piece, we finally get that f_0 and f_1 are topologically equivalent. This completes the proof of Theorem 1.

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Newton polygons and families of polynomials

NEWTON POLYGONS AND FAMILIES OF POLYNOMIALS

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Abstract. We consider a continuous family (f_s), $s \in [0,1]$ of complex polynomials in two variables with isolated singularities, that are Newton non-degenerate. We suppose that the Euler characteristic of a generic fiber is constant. We firstly prove that the set of critical values at infinity depends continuously on s, and secondly that the degree of the f_s is constant (up to an algebraic automorphism of $\mathbb{C}^2$).

1. Introduction

We consider a family $(f_s)_{s\in[0,1]}$ of complex polynomials in two variables with isolated singularities. We suppose that coefficients are continuous functions of s. For all s, there exists a finite *bifurcation set* $\mathcal{B}(s)$ such that the restriction of f_s above $\mathbb{C} \setminus \mathcal{B}(s)$ is a locally trivial fibration. It is known that $\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_\infty(s)$, where $\mathcal{B}_{\mathit{aff}}(s)$ is the set of *affine critical values*, that is to say the image by f_s of the critical points; $\mathcal{B}_\infty(s)$ is the set of *critical values at infinity*. For $c \notin \mathcal{B}(s)$, the Euler characteristic verifies $\chi(f_s^{-1}(c)) = \mu(s) + \lambda(s)$, where $\mu(s)$ is the *affine Milnor number* and $\lambda(s)$ is the *Milnor number at infinity*.

We will be interested in families such that the sum $\mu(s) + \lambda(s)$ is constant. These families are interesting in the view of μ-constant type theorem, see [HZ, HP, Ti, Bo, BT]. We say that a multi-valued function $s \mapsto F(s)$ is *continuous* if at each point $\sigma \in [0,1]$ and at each value $c(\sigma) \in F(\sigma)$ there is a neighborhood I of σ such that for all $s \in I$, there exists $c(s) \in F(s)$ near $c(\sigma)$. F is *closed*, if, for all points $\sigma \in [0,1]$, for all sequences $c(s) \in F(s)$, $s \neq \sigma$, such that $c(s) \to c(\sigma) \in \mathbb{C}$ as $s \to \sigma$, then $c(\sigma) \in F(\sigma)$. It it is well-known that $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is a continuous multi-valued function. But it is not necessarily closed: for example $f_s(x,y) = (x-s)(xy-1)$, then for $s \neq 0$, $\mathcal{B}_{\mathit{aff}}(s) = \{0, s\}$ but $\mathcal{B}_{\mathit{aff}}(0) = \varnothing$.

We will prove that $s \mapsto \mathcal{B}_\infty(s)$ and $s \mapsto \mathcal{B}(s)$ are closed continuous functions under some assumptions.

Theorem 1. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials such that $\mu(s)+\lambda(s)$ is constant and such that f_s is (Newton) non-degenerate for* *all $s \in [0,1]$, then the multi-valued function $s \mapsto \mathcal{B}_\infty(s)$ is continuous and closed.*

Remark. As a corollary we get the answer to a question of D. Siersma: is it possible to find a family with $\mu(s)+\lambda(s)$ constant such that $\lambda(0) > 0$ (equivalently $\mathcal{B}_\infty(0) \neq \varnothing$) and $\lambda(s) = 0$ (equivalently $\mathcal{B}_\infty = \varnothing$) for $s \in]0,1]$? Theorem 1 says that it is not possible for non-degenerate polynomials. Moreover for a family with $\mu(s) + \lambda(s)$ constant and $\lambda(s) > 0$ for $s \in]0,1]$ we have $\lambda(0) \geqslant \lambda(s) > 0$ by the (lower) semi-continuity of $\mu(s)$. In the case of a $\mathcal{FISI}$ deformation of polynomials of constant degree with a non-singular total space, the answer can be deduced from [ST, Theorem 5.4].

Remark. Theorem 1 does not imply that $\mu(s)$ and $\lambda(s)$ are constant. For example let the family $f_s(x,y) = x^2y^2 + sxy + x$. Then for $s = 0$, $\mu(0) = 0$, $\lambda(0) = 2$ with $\mathcal{B}_\infty(0) = \{0\}$, and for $s \neq 0$, $\mu(s) = 1$, $\lambda(s) = 1$ with $\mathcal{B}_{\mathit{aff}}(s) = \{0\}$ and $\mathcal{B}_\infty(s) = \{-\frac{s^2}{4}\}$.

The multi-valued function $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is continuous but not necessarily closed even if $\mu(s) + \lambda(s)$ is constant, for example (see [Ti]): $f_s(x,y) = x^4 - x^2y^2 + 2xy + sx^2$, then $\mu(s) + \lambda(s) = 5$. We have $\mathcal{B}_{\mathit{aff}}(0) = \{0\}$, $\mathcal{B}_\infty(0) = \{1\}$ and for $s \neq 0$, $\mathcal{B}_{\mathit{aff}} = \{0, 1 - \frac{s^2}{4}\}$, $\mathcal{B}_\infty(s) = \{1\}$. We notice that even if $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is not closed, the map $s \mapsto \mathcal{B}(s)$ is closed. This is expressed in the following corollary (of Theorems 1 and 3):

Corollary 2. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials such that $\mu(s) + \lambda(s)$ is constant and such that f_s is non-degenerate for all $s \in [0,1]$. Then the multi-valued function $s \mapsto \mathcal{B}(s)$ is continuous and closed.*

We are now interested in the constancy of the degree; in all hypotheses of global μ-constant theorems the degree of the f_s is supposed not to change (see [HZ, HP, Bo, BT]) and it is the only non-topological hypothesis. We prove that for non-degenerate polynomials in two variables the degree is constant except for a few cases, where the family is of quasi-constant degree. We will define in a combinatoric way in paragraph 3 what a family of *quasi-constant degree* is, but the main point is to know that such a family is of constant degree up to some algebraic automorphism of $\mathbb{C}^2$. More precisely, for each value $\sigma \in [0,1]$ there exists $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such $f_s \circ \Phi$ is of constant degree, for s in a neighborhood of σ. For example the family $f_s(x,y) = x + sy^2$ is of quasi-constant degree while the family $f_s(x,y) = sxy + x$ is not.

Theorem 3. *Let* $(f_s)_{s\in[0,1]}$ *be a family of complex polynomials such that* $\mu(s)+\lambda(s)$ *is constant and such that* f_s *is non-degenerate for all* $s\in]0,1]$, *then either* $(f_s)_{s\in[0,1]}$ *is of constant degree or* $(f_s)_{s\in[0,1]}$ *is of quasi-constant degree.*

Remark. In theorem 3, f_0 may be degenerate.

As a corollary we get a μ-constant theorem without hypothesis on the degree:

Theorem 4. *Let* $(f_s)_s \in [0,1]$ *be a family of polynomials in two variables with isolated singularities such that the coefficients are continuous function of* s. *We suppose that* f_s *is non-degenerate for* $s\in]0,1]$, *and that the integers* $\mu(s)+\lambda(s), \#\mathcal{B}(s)$ *are constant (*$s\in[0,1]$*) then the polynomials* f_0 *and* f_1 *are topologically equivalent.*

It is just the application of the μ-constant theorem of [Bo], [BT] to the family (f_s) or $(f_s\circ\Phi)$. Two kinds of questions can be asked : are Theorems 1 and 3 true for degenerate polynomials? are they true for polynomials in more than 3 variables? I would like to thank Prof. Günter Ewald for discussions concerning Theorem 3 in n variables (that unfortunately only yield that the given proof cannot be easily generalized).

2. Tools

2.1. **Definitions.** We will recall some basic facts about Newton polygons, see [Ko], [CN], [NZ]. Let $f\in\mathbb{C}[x,y]$, $f(x,y)=\sum_{(p,q)\in\mathbb{N}^2} a_{p,q}x^py^q$. We denote $\mathrm{supp}(f)=\{(p,q)\mid a_{p,q}\neq 0\}$, by abuse $\mathrm{supp}(f)$ will also denote the set of monomials $\{x^py^q\mid (p,q)\in\mathrm{supp}(f)\}$. $\Gamma_-(f)$ is the convex closure of $\{(0,0)\}\cup\mathrm{supp}(f)$, $\Gamma(f)$ is the union of closed faces which do not contain $(0,0)$. For a face γ, $f_\gamma=\sum_{(p,q)\in\gamma} a_{p,q}x^py^q$. The polynomial f is *(Newton) non-degenerate* if for all faces γ of $\Gamma(f)$ the system

$$\frac{\partial f_\gamma}{\partial x}(x,y)=0;\quad \frac{\partial f_\gamma}{\partial y}(x,y)=0$$

has no solution in $\mathbb{C}^*\times\mathbb{C}^*$.

We denote by S the area of $\Gamma_-(f)$, by a the length of the intersection of $\Gamma_-(f)$ with the x-axis, and by b the length of the intersection of $\Gamma_-(f)$ with the y-axis (see Figure 1). We define

$$\nu(f)=2S-a-b+1.$$

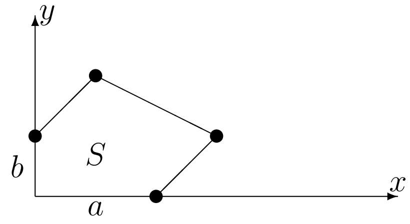

FIGURE 1. Newton polygon of f and $\nu(f) = 2S - a - b + 1$.

2.2. **Milnor numbers.** The following result is due to Pi. Cassou-Noguès [CN], it is an improvement of Kouchnirenko's result.

Theorem 5. *Let $f \in \mathbb{C}[x, y]$ with isolated singularities. Then*
(1) $\mu(f) + \lambda(f) \leqslant \nu(f)$.
(2) *If f is non-degenerate then $\mu(f) + \lambda(f) = \nu(f)$.*

2.3. **Critical values at infinity.** We recall the result of A. Némethi and A. Zaharia on how to estimate critical values at infinity. A polynomial $f \in \mathbb{C}[x, y]$ is *convenient for the x-axis* if there exists a monomial x^a in $\mathrm{supp}(f)$ $(a > 0)$; f is *convenient for the y-axis* if there exists a monomial y^b in $\mathrm{supp}(f)$ $(b > 0)$; f is *convenient* if it is convenient for the x-axis and the y-axis. It is well-known (see [Br]) that:

Lemma 6. *A non-degenerate and convenient polynomial with isolated singularities has no critical value at infinity: $\mathcal{B}_\infty = \varnothing$.*

Let $f \in \mathbb{C}[x, y]$ be a polynomial with $f(0, 0) = 0$ not depending only on one variable. Let γ_x and γ_y the two faces of $\Gamma_-(f)$ that contain the origin. If f is convenient for the x-axis then we set $\mathcal{C}_x = \varnothing$ otherwise γ_x is not included in the x-axis and we set

$$\mathcal{C}_x = \left\{ f_{\gamma_x}(x, y) \mid (x, y) \in \mathbb{C}^* \times \mathbb{C}^* \text{ and } \frac{\partial f_{\gamma_x}}{\partial x}(x, y) = \frac{\partial f_{\gamma_x}}{\partial y}(x, y) = 0 \right\}.$$

In a similar way we define $\mathcal{C}_y$.

A result of [NZ, Proposition 6] is:

Theorem 7. *Let $f \in \mathbb{C}[x, y]$ be a non-degenerate and non-convenient polynomial with $f(0, 0) = 0$, not depending only on one variable. The set of critical values at infinity of f is*

$$\mathcal{B}_\infty = \mathcal{C}_x \cup \mathcal{C}_y \quad \text{or} \quad \mathcal{B}_\infty = \{0\} \cup \mathcal{C}_x \cup \mathcal{C}_y.$$

Unfortunately this theorem does not determine whether $0 \in \mathcal{B}_\infty$ (and notice that the value 0 may be already included in $\mathcal{C}_x$ or $\mathcal{C}_y$). This value 0 is treated in the following lemma.

Lemma 8. *Let $f \in \mathbb{C}[x,y]$ be a non-degenerate and non-convenient polynomial, with isolated singularities and with $f(0,0)=0$. Then*

$$\mathcal{B}_\infty = \mathcal{B}_{\infty,x} \cup \mathcal{B}_{\infty,y}$$

where we define:

(1) *if f is convenient for the x-axis then $\mathcal{B}_{\infty,x} := \varnothing$;*
(2) *otherwise there exists $x^p y$ in* $\operatorname{supp}(f)$ *where $p \geqslant 0$ is supposed to be maximal;*
  (a) *If $x^p y$ is in a face of $\Gamma_-(f)$ then $\mathcal{B}_{\infty,x} := \mathcal{C}_x$ and $0 \notin \mathcal{B}_{\infty,x}$;*
  (b) *If $x^p y$ is not in a face of $\Gamma_-(f)$ then $\mathcal{B}_{\infty,x} := \{0\} \cup \mathcal{C}_x$;*
(3) *we set a similar definition for $\mathcal{B}_{\infty,y}$.*

Theorem 7 and its refinement Lemma 8 enable to calculate $\mathcal{B}_\infty$ from $\operatorname{supp}(f)$. The different cases of Lemma 8 are pictured in Figures 2 and 3.

Proof. As f is non-convenient with $f(0,0)=0$ we may suppose that f is non-convenient for the x-axis so that $f(x,y) = yk(x,y)$. But f has isolated singularities, so y does not divide k. Then there is a monomial $x^p y \in \operatorname{supp}(f)$, we can suppose that $p \geqslant 0$ is maximal among monomials $x^k y \in \operatorname{supp}(f)$.

Let $d = \deg f$. Let $\bar{f}(x,y,z) - cz^d$ be the homogeneization of $f(x,y) - c$; at the point at infinity $P = (1:0:0)$, we define $g_c(y,z) = \bar{f}(1,y,z) - cz^d$. Notice that only $(1:0:0)$ and $(0:1:0)$ can be singularities at infinity for f. The value 0 is a critical value at infinity for the point at infinity P (that is to say $0 \in \mathcal{B}_{\infty,x}$) if and only if $\mu_P(g_0) > \mu_P(g_c)$ where c is a generic value.

The Newton polygon of the germ of singularity g_c can be computed from the Newton polygon $\Gamma(f)$, for $c \neq 0$, see [NZ, Lemma 7]. If A, B, O are the points on the Newton diagram of coordinates $(d,0), (0,d), (0,0)$, then the Newton diagram of g_c has origin A with y-axis AB, z-axis AO, and the convex closure of $\operatorname{supp}(g_c)$ corresponds to $\Gamma_-(f)$.

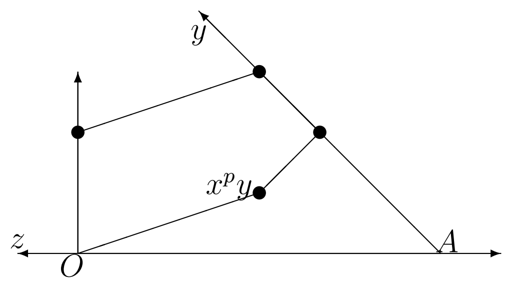

FIGURE 2. Newton polygon of g_c. First case: $0 \notin \mathcal{B}_{\infty,x}$.

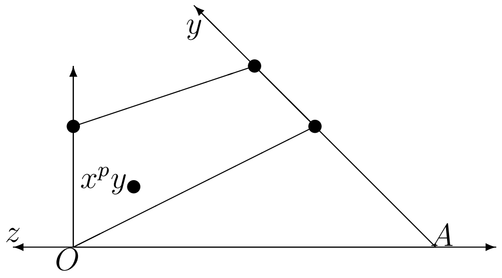

FIGURE 3. Newton polygon of g_c. Second case: $0 \in \mathcal{B}_{\infty,x}$.

We denote by Δ_c the Newton polygon of the germ g_c, for a generic value c, Δ_c is non-degenerate and $\mu_P(g_c) = \nu(\Delta_c)$. The Newton polygon Δ_0 has no common point with the z-axis AO but ν may be defined for non-convenient series, see [Ko, Definition 1.9].

If $x^p y$ is in the face γ_x of $\Gamma_-(f)$ then Δ_0 is non-degenerate and $\nu(\Delta_0) = \nu(\Delta_c)$, then by [Ko, Theorem 1.10] $\mu_P(g_0) = \nu(\Delta_0)$ and $\mu_P(g_c) = \nu(\Delta_c)$. So $\mu_P(g_0) = \mu_P(g_c)$ and 0 is not a critical value at infinity for the point $P : 0 \notin \mathcal{B}_{\infty,x}$.

If $x^p y$ is not in a face of $\Gamma_-(f)$ then there is a triangle Δ_c that disappears in Δ_0, by the positivity of ν (see below) we have $\nu(\Delta_0) > \nu(\Delta_c)$, then by [Ko, Theorem 1.10]: $\mu_P(g_0) \geqslant \nu(\Delta_0) > \nu(\Delta_c) = \mu_P(g_c)$. So we have $0 \in \mathcal{B}_{\infty,x}$. □ □

2.4. **Additivity and positivity.** We need a variation of Kouchnirenko's number ν. Let T be a polytope whose vertices are in $\mathbb{N} \times \mathbb{N}$, $S > 0$ the area of T, a the length of the intersection of T with the x-axis, and b the length of the intersection of T with the y-axis. We define

$$\tau(T) = 2S - a - b, \text{ so that, } \nu(T) = \tau(T) + 1.$$

It is clear that τ is additive: $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2) - \tau(T_1 \cap T_2)$, and in particular if $T_1 \cap T_2$ has null area then $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2)$. This formula enables us to argue on triangles only (after a triangulation of T).

Let T_0 be the triangle defined by the vertices $(0,0), (1,0), (0,1)$, we have $\nu(T_0) = -1$. We have the following facts, for every triangle $T \neq T_0$:

(1) $\nu(T) \geqslant 0$;
(2) $\nu(T) = 0$ if and only if T has an edge contained in the x-axis or the y-axis and the height of T (with respect to this edge) is 1.

Remark. The formula of additivity can be generalized in the n-dimensional case, but the positivity can not. Here is a counter-example found by Günter Ewald: Let $n = 4$, a a positive integer and let T be the polytope whose vertices are: $(1, 0, 0, 0)$, $(1 + a, 0, 0, 0)$, $(1, 1, 1, 0)$, $(1, 2, 1, 0)$, $(1, 1, 1, 1)$ then $\tau(T) = \nu(T) + 1 = -a < 0$.

2.5. **Families of polytopes.** We consider a family $(f_s)_{s\in[0,1]}$ of complex polynomials in two variables with isolated singularities. We suppose that $\mu(s)+\lambda(s)$ remains constant. We denote by $\Gamma(s)$ the Newton polygon of f_s. We suppose that f_s is non-degenerate for $s \in]0, 1]$.

We will always assume that the only critical parameter is $s = 0$. We will say that a monomial $x^p y^q$ *disappears* if $(p, q) \in \text{supp}(f_s) \setminus \text{supp}(f_0)$ for $s \neq 0$. By extension a triangle of $\mathbb{N} \times \mathbb{N}$ disappears if one of its vertices (which is a vertex of $\Gamma(s)$, $s \neq 0$) disappears. Now after a triangulation of $\Gamma(s)$ we have a finite number of triangles T that disappear (see Figure 4, on pictures of the Newton diagram, a plain circle is drawn for a monomial that does not disappear and an empty circle for monomials that disappear).

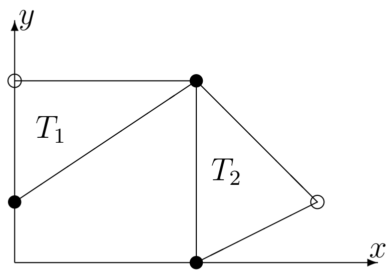

Figure 4. Triangles that disappear.

We will widely use the following result, under the hypotheses of Theorem 3:

Lemma 9. *Let $T \neq T_0$ be a triangle that disappears then $\tau(T) = 0$.*

Proof. We suppose that $\tau(T) > 0$. By the additivity and positivity of τ we have for $s \in]0, 1]$:

$$\nu(s) = \nu(\Gamma(s)) \geqslant \nu(\Gamma(0)) + \tau(T) > \nu(0).$$

Then by Theorem 5,

$$\mu(s) + \lambda(s) = \nu(s) > \nu(0) \geqslant \mu(0) + \lambda(0).$$

This gives a contradiction with $\mu(s) + \lambda(s) = \mu(0) + \lambda(0)$.

We remark that we do not need f_0 to be non-degenerate because in all cases we have $\nu(0) \geqslant \mu(0) + \lambda(0)$. □ □

3. Constancy of the degree

3.1. **Families of quasi-constant degree.** Let $\sigma \in [0,1]$, we choose a small enough neighborhood I of σ. Let $\mathcal{M}_\sigma$ be the set of monomials that disappear at σ: $\mathcal{M}_\sigma = \operatorname{supp}(f_s) \setminus \operatorname{supp}(f_\sigma)$ for $s \in I \setminus \{\sigma\}$. The family $(f_s)_{s\in[0,1]}$ is of *quasi-constant degree at* σ if

$$\text{there exists } x^p y^q \in \operatorname{supp}(f_\sigma) \text{ such that}$$
$$(\forall x^{p'}y^{q'} \in \mathcal{M}_\sigma \quad (p > p') \text{ or } (p = p' \text{ and } q > q'))$$
$$\text{or } (\forall x^{p'}y^{q'} \in \mathcal{M}_\sigma \quad (q > q') \text{ or } (q = q' \text{ and } p > p')).$$

The family $(f_s)_{s\in[0,1]}$ is of *quasi-constant degree* if it is of quasi-constant degree at each point σ of $[0,1]$. The terminology is justified by the following remark:

Lemma 10. *If (f_s) is of quasi-constant degree at $\sigma \in [0,1]$, then there exists $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such that $\deg f_s \circ \Phi$ is constant in a neighborhood of σ.*

The proof is simple: suppose that $x^p y^q$ is a monomial of $\operatorname{supp}(f_\sigma)$ such that for all $x^{p'}y^{q'} \in \mathcal{M}_\sigma$, $p > p'$ or ($p = p'$ and $q > q'$). We set $\Phi(x,y) = (x + y^\ell, y)$ with $\ell \gg 1$. Then the monomial of highest degree in $f_s \circ \Phi$ is $y^{q+p\ell}$ and does not disappear at σ. For example let $f_s(x,y) = xy + sy^3$, we set $\Phi(x,y) = (x+y^3, y)$ then $f_s \circ \Phi(x,y) = y^4 + xy + sy^3$ is of constant degree.

We prove Theorem 3. We suppose that the degree changes, more precisely we suppose that $\deg f_s$ is constant for $s \in]0,1]$ and that $\deg f_0 < \deg f_s$, $s \in]0,1]$. As the degree changes the Newton polygon $\Gamma(s)$ cannot be constant, that means that at least one vertex of $\Gamma(s)$ disappears.

3.2. **Exceptional case.** We suppose that f_0 is a one-variable polynomial, for example $f_0 \in \mathbb{C}[y]$. As f_0 has isolated singularities then $f_0(x,y) = a_0 y + b_0$, so $\mu(0) = \lambda(0) = 0$, then for all s, $\mu(s) = \lambda(s) = 0$. So $\nu(s) = \nu(\Gamma(s)) = 0$, then $\deg_y f_s = 1$, and $f_s(x,y) = a_s y + b_s(x)$, so $(f_s)_{s\in[0,1]}$ is a family of quasi-constant degree (see Figure 5). We exclude this case for the end of the proof.

3.3. **Case to exclude.** We suppose that a vertex $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears. Then there exists a triangle T that disappears whose faces are not contained in the axis. Then $\tau(T) > 0$ that contradicts Lemma 9 (see Figure 6).

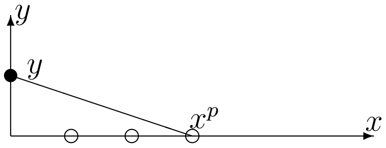

FIGURE 5. Case $f_0 \in \mathbb{C}[y]$.

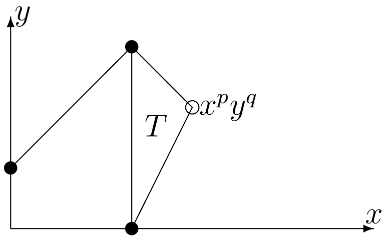

FIGURE 6. Case where a monomial $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears.

3.4. **Case where a monomial x^a or y^b disappears (but not both).** If, for example the monomial y^b of $\Gamma(s)$ disappears and x^a does not, then we choose a monomial $x^p y^q$, with maximal p, among monomials in $\mathrm{supp}(f_s)$. Certainly $p \geqslant a > 0$. We also suppose that q is maximal among monomials $x^p y^k \in \mathrm{supp}(f_s)$. If $q = 0$ then $p = a$, and the monomial $x^p y^q = x^a$ does not disappear (by assumption). If $q > 0$ then $x^p y^q$ cannot disappear (see above). In both cases the monomial $x^p y^q$ proves that (f_s) is of quasi-constant degree.

3.5. **Case where both x^a and y^b disappear.**
Sub-case : No monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$. Then there is an area T with $\tau(T) > 0$ that disappears (see Figure 7). Contradiction.

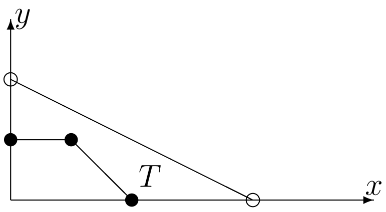

FIGURE 7. Sub-case : no monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$.

Sub-case : there exists a monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$. We know that $x^p y^q$ is in $\Gamma(0)$ because it cannot disappear. As $\deg f_0 < \deg f_s$, a monomial $x^p y^q$ that does not disappear verifies $\deg x^p y^q = p + q <$

$\deg f_s$, ($s \in]0,1]$). So the monomial of highest degree is x^a or y^b. We will suppose that it is y^b, so $d = b$, and the monomial y^b disappears. Let $x^p y^q$ be a monomial of $\Gamma(s)$, $p, q > 0$ with minimal q. By assumption such a monomial exists. Then certainly we have $q = 1$, otherwise there exists a region T that disappears with $\tau(T) > 0$ (on Figure 8 the regions T_1 and T_2 verify $\nu(T_1) = 0$ and $\nu(T_2) = 0$). For the same reason the monomial $x^{p'} y^{q'}$ with minimal p' verifies $p' = 1$.

We look at the segments of $\Gamma(s)$, starting from $y^b = y^d$ and ending at x^a. The first segment is from y^d to $xy^{q'}$, ($p' = 1$) and we know that $p' + q' < d$ so the slope of this segment is strictly less than -1. By the convexity of $\Gamma(s)$ all the following slopes are strictly less than -1. The last segment is from $x^p y$ to x^a, with a slope strictly less than -1, so $a \leqslant p$. Then the monomial $x^p y$ gives that $(f_s)_{s \in [0,1]}$ is of quasi-constant degree.

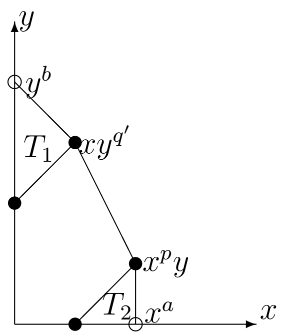

FIGURE 8. Sub-case : existence of monomials $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$.

4. CONTINUITY OF THE CRITICAL VALUES

We now prove Theorem 1. We will suppose that $s = 0$ is the only problematic parameter. In particular $\Gamma(s)$ is constant for all $s \in]0,1]$.

4.1. **The Newton polygon changes.** That is to say $\Gamma(0) \neq \Gamma(s)$, $s \neq 0$. As in the proof of Theorem 3 (see paragraph 3) we remark:

- If f_0 is a one-variable polynomial then $\mathcal{B}_\infty(s) = \varnothing$ for all $s \in [0,1]$.
- A vertex $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ cannot disappear.

So we suppose that a monomial x^a of $\Gamma(s)$ disappears (a similar proof holds for y^b). Then for $s \in]0,1]$ the monomial x^a is in $\Gamma(s)$, so there are no critical values at infinity for f_s at the point $P = (1:0:0)$. If $\Gamma(0)$ contains a monomial $x^{a'}$, $a' > 0$ then there are no critical values at infinity for f_0 at the point P . So we suppose that all monomials x^k disappear. Then a monomial $x^p y^q$ of $\operatorname{supp}(f_0)$ with minimal $q > 0$, verifies $q = 1$, otherwise there would exist a region T with $\tau(T) > 0$ (in contradiction with the constancy of $\mu(s) + \lambda(s)$, see Lemma 9). And for the same reason if we choose $x^p y$ in $\operatorname{supp}(f_0)$ with maximal p then $p > 0$ and $x^p y \in \Gamma(0)$. Now the edge of $\Gamma_-(f_0)$ that contains the origin and the monomial $x^p y$ (with maximal p) begins at the origins and ends at $x^p y$ (so in particular there is no monomial $x^{2p}y^2$, $x^{3p}y^3$ in $\operatorname{supp}(f_0)$). Now from Theorem 7 and Lemma 8 we get that there are no critical values at infinity for f_0 at P.

So in case where $\Gamma(s)$ changes, we have for all $s \in [0,1]$, $\mathcal{B}_\infty(s) = \varnothing$.

4.2. **The Newton polygon is constant : case of non-zero critical values.** We now prove the following lemma that ends the proof of Theorem 1.

Lemma 11. *Let a family $(f_s)_{s\in[0,1]}$ such that f_s is non-degenerate for all $s \in [0,1]$ and $\Gamma(s)$ is constant, then the multi-valued function $s \mapsto \mathcal{B}_\infty(s)$ is continuous and closed.*

In this paragraph and the next one we suppose that $f_s(0,0) = 0$, that is to say the constant term of f_s is zero. We suppose that $c(0) \in \mathcal{B}_\infty(0)$ and that $c(0) \neq 0$. Then $c(0)$ has been obtained by the result of Néméthi-Zaharia (see Theorem 7). There is a face γ of $\Gamma_-(f_0)$ that contains the origin such that $c(0)$ is in the set:

$$\mathcal{C}_\gamma(0) = \left\{ (f_0)_\gamma(x,y) \mid (x,y) \in (\mathbb{C}^*)^2 \text{ and } \frac{\partial (f_0)_\gamma}{\partial x}(x,y) = \frac{\partial (f_0)_\gamma}{\partial y}(x,y) = 0 \right\}.$$

Now, as $\Gamma(s)$ is constant, γ is a face of $\Gamma_-(s)$ for all s. There exists a family of polynomials $h_s \in \mathbb{C}[t]$ and a monomial $x^p y^q$ ($p, q > 0$, $\gcd(p,q) = 1$) such that $(f_s)_\gamma(x,y) = h_s(x^p y^q)$. The family (h_s) is continuous (in s) and is of constant degree (because $\Gamma(s)$ is constant). The set $\mathcal{C}_\gamma(0)$ and more generally the set $\mathcal{C}_\gamma(s)$ can be computed by

$$\mathcal{C}_\gamma(s) = \left\{ h_s(t) \mid t \in \mathbb{C}^* \text{ and } h_s'(t) = 0 \right\}.$$

As $c(0) \in \mathcal{C}_\gamma(0)$ there exists a $t_0 \in \mathbb{C}^*$ with $h_0'(t_0) = 0$, and for s near 0 there is a $t_s \in \mathbb{C}^*$ near t_0 with $h_s'(t_s) = 0$ (because $h_s'(t)$ is a continuous function of s of constant degree in t). Then $c(s) = h_s(t_s)$ is a critical value at infinity near $c(0)$ and we get the continuity.

4.3. **The Newton polygon is constant : case of the value 0.** We suppose that $c(0) = 0 \in \mathcal{B}_\infty(0)$ and that $f_s(x,y) = yk_s(x,y)$. We will deal with the point at infinity $P = (1:0:0)$, the point $(0:1:0)$

is treated in a similar way. Let $x^p y$ be a monomial of $\mathrm{supp}(f_s)$ with maximal $p \geqslant 0$, $s \neq 0$. If $x^p y$ is not in a face of $\Gamma(s)$ then $0 \in \mathcal{B}_\infty(s)$ for all $s \in [0,1]$, and we get the continuity. Now we suppose that $x^p y$ is in a face of $\Gamma(s)$; then $x^p y$ disappears otherwise 0 is not a critical value at infinity (at the point P) for all $s \in [0,1]$. As $\Gamma(s)$ is constant then the face γ that contains the origin and $x^p y$ for $s \neq 0$ is also a face of $\Gamma(0)$, then there exists a monomial $(x^p y)^k$, $k > 1$ in $\mathrm{supp}(f_0)$. Then $(f_s)_\gamma = h_s(x^p y)$, $h_s \in \mathbb{C}[t]$. We have $\deg h_s > 1$, with $h_s(0) = 0$ (because $f(0,0) = 0$) and $h'_0(0) = 0$ (because $x^p y$ disappears). Then $0 \in \mathcal{C}_\gamma(0) \subset \mathcal{B}_\infty(0)$ but by continuity of h_s we have a critical value $c(s) \in \mathcal{C}_\gamma(s) \subset \mathcal{B}_\infty(s)$ such that $c(s)$ tends towards 0 (as $s \to 0$). It should be noticed that for $s \neq 0$, $c(s) \neq 0$.

In all cases we get the continuity of $\mathcal{B}_\infty(s)$.

4.4. **Proof of the closeness of** $s \mapsto \mathcal{B}_\infty(s)$**.** We suppose that $c(s) \in \mathcal{B}_\infty(s)$, is a continuous function of $s \neq 0$, with a limit $c(0) \in \mathbb{C}$ at $s = 0$. We have to prove that $c(0) \in \mathcal{B}_\infty(0)$. As there are critical values at infinity we suppose that $\Gamma(s)$ is constant.
Case $c(0) \neq 0$. Then for s near 0, $c(s) \neq 0$ by continuity, then $c(s)$ is obtained as a critical value of $h_s(t)$. By continuity $c(0)$ is a critical value of $h_0(t)$: $h_0(t_0) = c(0)$, $h'_0(t_0) = 0$; as $c(0) \neq 0$, $t_0 \neq 0$ (because $h_0(0) = 0$). Then $c(0) \in \mathcal{B}_\infty(0)$.
Case $c(0) = 0$. Then let $x^p y$ be the monomial of $\mathrm{supp}(f_s)$, $s \neq 0$, with maximal p. By Lemma 8 if $x^p y \notin \Gamma(s)$ for $s \in]0,1]$ then $0 \in \mathcal{B}_\infty(s)$ for all $s \in [0,1]$ and we get closeness. If $x^p y \in \Gamma(s)$, $s \neq 0$, then as $c(s) \to 0$ we have that $x^p y$ disappears, so $x^p y \notin \Gamma(0)$, then by Lemma 8, $c(0) = 0 \in \mathcal{B}_\infty(0)$.

4.5. **Proof of the closeness of** $s \mapsto \mathcal{B}(s)$**.** We now prove Corollary 2. The multi-valued function $s \mapsto \mathcal{B}(s)$ is continuous because $\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_\infty(s)$ and $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$, $s \mapsto \mathcal{B}_\infty(s)$ are continuous. For closeness, it remains to prove that if $c(s) \in \mathcal{B}_{\mathit{aff}}(s)$ is a continuous function with a limit $c(0) \in \mathbb{C}$ at $s = 0$ then $c(0) \in \mathcal{B}(0)$.

We suppose that $c(0) \notin \mathcal{B}_{\mathit{aff}}(0)$. There exist critical points $Q_s = (x_s, y_s) \in \mathbb{C}^2$ of f_s with $f_s(x_s, y_s) = c(s)$, $s \neq 0$. We can extract a countable set $\mathcal{S}$ of $]0,1]$ such that the sequence $(Q_s)_{s \in \mathcal{S}}$ converges towards P in $\mathbb{C}P^2$. As $c(0) \notin \mathcal{B}_{\mathit{aff}}(0)$ we have that P relies on the line at infinity and we may suppose that $P = (0:1:0)$.

By Theorem 3 we may suppose, after an algebraic automorphism of $\mathbb{C}^2$ if necessary, that $d = \deg f_s$ is constant. Now we look at $g_{s,c}(x,z) = \bar{f}_s(x,1,z) - cz^d$. The critical point Q_s of f_s with critical value $c(s)$ gives a critical point $Q'_s = (\frac{x_s}{y_s}, \frac{1}{y_s})$ of $g_{s,c(s)}$ with critical value 0 (see

[Bo, Lemma 21]). Then by semi-continuity of the local Milnor number on the fiber $g_{s,c(s)}^{-1}(0)$ we have $\mu_P(g_{0,c(0)}) \geqslant \mu_P(g_{s,c(s)}) + \mu_{Q'_s}(g_{s,c(s)}) > \mu_P(g_{s,c(s)})$. As $\mu(s) + \lambda(s)$ is constant we have $\mu_P(g_{s,c})$ constant for a generic c (see [ST, Corollary 5.2] or [BT]). Then we have $\mu_P(g_{0,c(0)}) - \mu_P(g_{0,c}) > \mu_P(g_{s,c(s)}) - \mu_P(g_{s,c}) \geqslant 0$. Then $c(0) \in \mathcal{B}_\infty(0)$. And we get closeness for $s \mapsto \mathcal{B}(s)$.

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Meromorphic functions, bifurcation sets and fibred links

avec Anne Pichon

MEROMORPHIC FUNCTIONS, BIFURCATION SETS AND FIBRED LINKS

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Abstract. We give a necessary condition for a meromorphic function in several variables to give rise to a Milnor fibration of the local link (respectively of the link at infinity). In the case of two variables we give some necessary and sufficient conditions for the local link (respectively the link at infinity) to be fibred.

1. Introduction

A famous result of J. Milnor [11] states that the link $f^{-1}(0) \cap \mathbb{S}^{2n-1}_\varepsilon$ $(0 < \varepsilon \ll 1)$ of a holomorphic germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is a fibred link and moreover that a fibration is given by the so-called Milnor fibration $\frac{f}{|f|} : \mathbb{S}^{2n-1}_\varepsilon \setminus f^{-1}(0) \longrightarrow \mathbb{S}^1$. Throughout this paper $\mathbb{S}^{n-1}_r$ denotes the sphere with radius r centered at the origin of $\mathbb{R}^n$.

The proof of this result has been extended in several directions in order to construct some natural fibrations in other situations of singularity theory. In this paper, we focus on two of them :

(1) Let $f : (\mathbb{R}^{n+k}, 0) \longrightarrow (\mathbb{R}^k, 0)$ be a real analytic germ with an isolated critical point at the origin. J. Milnor [11, Chapter 11] proved that for every sufficiently small sphere $\mathbb{S}^{n+k-1}_\varepsilon$ centered at the origin in $\mathbb{R}^{n+k}$, the complement $\mathbb{S}^{n+k-1}_\varepsilon \setminus L_f$ of the link $L_f = \mathbb{S}^{n+k-1}_\varepsilon \cap f^{-1}(0)$ fibres over the circle. As pointed out by Milnor, the fibration is not necessarily given by the Milnor map $\frac{f}{\|f\|}$.

This result can be extended to a real analytic germ $f : (X, p) \longrightarrow (\mathbb{R}^k, 0)$ with isolated critical value satisfying a suitable stratification condition, where (X, p) is a germ of real analytic space with isolated singularity at p ([15], Theorem 1.1).

(2) Another direction deals with links at infinity. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map. The link at infinity associated with the fibre $f^{-1}(0)$ is defined by $L_{f,\infty} = f^{-1}(0) \cap \mathbb{S}^{2n-1}_R$ for a sufficiently large radius $R \gg 1$. In [12], A. Némethi and A. Zaharia proved that under a condition called *semitame*, the link at infinity $L_{f,\infty}$ is fibred by the Milnor map $\frac{f}{|f|} : \mathbb{S}^{2n-1}_R \setminus L_{f,\infty} \longrightarrow \mathbb{S}^1$.

When $n = 2$, A. Bodin in [1] proved that the link at infinity is fibred if and only if the semitame condition holds, or equivalently iff the set of critical values at infinity is void or equal to $\{0\}$.

In the first part of the paper we consider a meromorphic function f/g, where $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ and $g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ are two holomorphic germs. We firstly define a condition called *semitame at the origin* for f/g by adapting the definition of [12]. Namely it consists of defining a bifurcation set B which reflects the behaviour of the points of non-transversality between the fibres of f/g and the spheres $\mathbb{S}^{2n-1}_\varepsilon$, $\varepsilon \ll 1$, centered at the origin of $\mathbb{C}^n$. In particular, under the semitame condition $B = \{0, \infty\}$, there is no non-isolated singular points of the meromorphic function f/g (nor points of indeterminacy) outside $(f = 0) \cup (g = 0)$.

Theorem 1.1. *Let $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors such that the meromorphic germ $\frac{f}{g}$ verifies the semitame condition $B = \{0, \infty\}$ at the origin. Then for all sufficiently small $\varepsilon > 0$ the Milnor map*

$$\frac{f/g}{|f/g|} : \mathbb{S}^{2n-1}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration.

Moreover for $n = 2$ this fibration is a fibration of the link $L_{f/g} = L_f \cup -L_g$.

The second part is devoted to find a reciprocal of Theorem 1.1 in the case of two variables: $n = 2$. More precisely we give other equivalent conditions to the equivalence (1) $\Leftrightarrow$ (2) due to A. Pichon and J. Seade.

Theorem 1.2. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs without common branches. Then the following conditions are equivalent:*

1. *The link $L_f \cup -L_g$ is fibred;*
2. *The real analytic germ $f\overline{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has 0 as an isolated critical value;*
3. *The Milnor map $\frac{f\overline{g}}{|f\overline{g}|} = \frac{f/g}{|f/g|} : \mathbb{S}^3_\varepsilon \setminus (L_f \cup L_g)$ is a C^∞ locally trivial fibration;*
4. *The meromorphic map f/g holds the semitame condition at the origin;*
5. *Each $c \neq 0, \infty$ is a generic value of the local pencil generated by f and g.*

The equivalence (1) $\Leftrightarrow$ (2) is proved by A. Pichon in [14] when f and g have 0 as an isolated critical point and generalized by A. Pichon and J. Seade in [15]. In particular Theorem 1.2 gives an alternative proof to (1) $\Rightarrow$ (5) which was first observed by F. Michel and H. Maugendre in [10].

Notice that a natural question consists of comparing the Milnor fibration $\frac{f\bar{g}}{|f\bar{g}|} : \mathbb{S}^3_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1$ with the local fibrations of meromorphic germs introduced in [4, 5]; this will be done in a forthcoming work with J. Seade.

The last part of the paper is devoted to the global case. Let $f : \mathbb{C}^2 \to \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ be two polynomial maps. The link at infinity $L_{f/g,\infty}$ of the meromorphic function f/g is defined by $L_{f/g,\infty} = L_{f,\infty} \cup -L_{g,\infty}$. In [7], after observing that $\frac{f\bar{g}}{|f\bar{g}|} = \frac{f/g}{|f/g|}$, M. Hirasawa and L. Rudolph ask whether the methods developed in [14] can be adapted to f/g at infinity. Namely there are two natural questions: Under which conditions is the link $L_{f/g,\infty}$ fibred? When $L_{f/g,\infty}$ is fibred, is the Milnor map $\frac{f/g}{|f/g|} : \mathbb{S}^3_R \setminus L_{f/g,\infty} \to \mathbb{S}^1$ a fibration of $L_{f/g,\infty}$?

We define a semitame condition at infinity for meromorphic maps, that enables one to adapt the methods of [12] and [1] and of the first sections of this work. We first get a version of Theorem 1.1 at infinity (see Theorem 3.4) and then we obtain the following result, which is a complete answer to the question of M. Hirasawa and L. Rudolph:

Theorem 1.3. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ be two polynomial maps. The following conditions are equivalent:*

(1) *the meromorphic function f/g is semitame at infinity;*
(2) *the link $L_{f/g,\infty}$ is fibred.*

Moreover, if these conditions hold, then the Milnor map $\frac{f/g}{|f/g|} : \mathbb{S}^3_R \setminus L_{f/g,\infty} \to \mathbb{S}^1$ is a fibration of the link $L_{f/g,\infty}$.

It should be noticed that the local situation for meromorphic maps is similar to the polynomial situation at infinity and the situation at infinity for meromorphic maps is morally the gluing of a finite number of meromorphic local situations.

2. Milnor map of a meromorphic function: the local case

The multilink of a meromorphic function. An *oriented link* in $\mathbb{S}^k$ is a disjoint union of oriented $(k-2)$-spheres $K_1, \ldots, K_\ell$ embedded in $\mathbb{S}^k$. A multilink is the data of an oriented link $L = K_1 \cup \ldots \cup K_\ell$ together with a multiplicity $n_i \in \mathbb{Z}$ associated to each component K_i of L. We denote $L = n_1 K_1 \cup \ldots \cup n_\ell K_\ell$ with the convention $n_i K_i = (-n_i)(-K_i)$ where $-K_i$ means K_i with the opposite orientation (see [3]).

For example, the multilink of a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is defined by $L_f = n_1 L_{f_1} \cup \ldots \cup n_\ell L_{f_\ell}$, where $f = \prod_{i=1}^{\ell} f_i^{n_i}$ is the decomposition of f in the UFD ring $\mathbb{C}\{x_1, \ldots, x_n\}$ and where $L_{f_i} = f_i^{-1}(0) \cap \mathbb{S}_\varepsilon^{2n-1}$, $\varepsilon \ll 1$, is oriented as the boundary of the piece of complex manifold $f_i^{-1}(0) \cap \mathbb{B}_\varepsilon^{2n}$. The diffeomorphism class of the pair $(\mathbb{S}_\varepsilon^{2n-1}, L_f)$ is independent of ε when ε is sufficiently small.

Definition 2.1. Let $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors in the decomposition in the UFD ring $\mathbb{C}\{x_1, \ldots, x_n\}$. Let $f, g : U \to \mathbb{C}$ be some representative of f and g. The *multilink* $L_{f/g}$ of the meromorphic function $f/g : U \longrightarrow \mathbb{P}^1$ is defined by

$$L_{f/g} = L_f \cup -L_g$$

where L_f and L_g denote the multilinks of f and g.

Semitame map at the origin. Let U be an open neighbourhood of 0 in $\mathbb{C}^n$ and let $f, g : U \longrightarrow \mathbb{C}$ be two holomorphic functions without common factors such that $f(0) = g(0) = 0$.

Let us consider the meromorphic function $f/g : U \to \mathbb{P}^1$ defined by $(f/g)(x) = [f(x) : g(x)]$. Notice that f/g is not defined on the whole U; its indetermination locus is $I(f/g) = \{x \in U \mid f(x) = 0 \text{ and } g(x) = 0\}$.

Adapting Milnor's definition let us define the gradient of f/g outside $I(f/g)$ by:

$$\operatorname{grad}(f/g) = \left(\overline{\frac{\partial(f/g)}{\partial x_1}}, \ldots, \overline{\frac{\partial(f/g)}{\partial x_n}} \right).$$

Let us consider the set

$$M(f/g) = \left\{ x \in U \setminus I(f/g) \mid \exists \lambda \in \mathbb{C}, \operatorname{grad} \frac{f}{g}(x) = \lambda x \right\}$$

consisting of the points of non-transversality between the fibres of f/g and the spheres $\mathbb{S}_r^{2n-1}$ centered at the origin of $\mathbb{C}^n$.

We define a bifurcation set $B \subset \mathbb{P}^1$ for the meromorphic function f/g as follows.

Definition 2.2. The bifurcation set B consists of all values $c \in \mathbb{P}^1$ such that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of points of $M(f/g)$ such that

$$\lim_{k \to \infty} x_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{f(x_k)}{g(x_k)} = c.$$

By convention and to avoid discussion we set

$$\{0, \infty\} \subset B.$$

The following definition is adapted from that of [12] which concerned polynomial maps at infinity.

Definition 2.3. The meromorphic germ f/g is *semitame at the origin* if $B = \{0, \infty\}$.

Remark 2.4.

(1) When f/g is semitame at the origin, the non isolated singular points of the meromorphic function f/g in an open neighbourhood of the origin belong to $(f = 0) \cup (g = 0)$. Indeed, assume that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of points of $U \setminus I(f/g)$ such that

$$\lim_{k \to \infty} x_k = 0 \text{ and } \operatorname{grad}(f/g)(x_k) = 0.$$

Then for all k, $x_k \in M(f/g)$. As the critical values of f/g are isolated, one can assume that there exists $c \in \mathbb{P}^1$ such that for all k, $(f/g)(x_k) = c$. The semitame condition therefore implies $c = 0$ or $c = \infty$.

(2) Notice that f and g can have non-isolated singularities, whereas f/g is semitame at the origin. See example 2.5.

(3) It is not hard to prove that the bifurcation set of g/f is the set of the inverse elements $\frac{1}{c}$ of the elements c of the bifurcation set of f/g.

(4) One can prove (using e.g. the arguments of the proof of Lemma 2.7) that a sequence (x_k) as in Definition 2.3 verifies:

$$\lim_{k \to \infty} \|x_k\| \cdot \|\operatorname{grad} \frac{f}{g}(x_k)\| = 0.$$

Example 2.5. (1) Let $f(x, y) = x^2$ and $g(x, y) = y^3$. Then $B = \{0, \infty\}$ and f/g is semitame at the origin.

(2) Let $f(x, y) = x(x + y^2) + y^3$ and $g(x, y) = y^3$. Then $B = \{0, 1, \infty\}$ and f/g is not semitame at the origin.

Fibration theorem under the semitame condition.

Theorem 2.6. *Let* $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ *be two germs of holomorphic functions without common factors such that the meromorphic germ* $\frac{f}{g}$ *is semitame at the origin. Then there exists* $0 < \varepsilon_0 \ll 1$ *such that for each* $\varepsilon \leqslant \varepsilon_0$ *the Milnor map*

$$\frac{f/g}{|f/g|} : \mathbb{S}^{2n-1}_{\varepsilon} \setminus L_{f/g} \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration.

The proof follows Milnor's proof [11, Chapter 4] with minor modifications. See also [12]. The main modification concerns Lemma 4.4 of [11], for which we give an adapted formulation and a detailed proof:

Lemma 2.7. *Assume that the meromorphic germ $G = f/g$ is semitame at the origin. Let $p : [0,1] \longrightarrow \mathbb{C}^n$ be a real analytic path with $p(0) = 0$ such that for all $t > 0$, $G(p(t)) \notin \{0, \infty\}$ and such that the vector* $\operatorname{grad} \log G(p(t))$ *is a complex multiple $\lambda(t)p(t)$ of $p(t)$. Then the argument of the complex number $\lambda(t)$ tends to 0 or π as $t \to 0$.*

Proof. The equality $\operatorname{grad} \log G(p(t)) = \lambda(t)p(t)$ implies $\operatorname{grad} G(p(t)) = \lambda(t)p(t)\overline{G(p(t))}$. Therefore $p(t) \in M(G)$.

Let us consider the expansions

$$\begin{aligned} p(t) &= \mathbf{a}t^\alpha + \ldots, \\ G(p(t)) &= bt^\beta + \ldots, \\ \operatorname{grad} G(p(t)) &= \mathbf{c}t^\gamma + \ldots, \end{aligned}$$

with $\alpha \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{Z}$, $\mathbf{a} \neq 0$, $b \neq 0$.

Assume that $\mathbf{c} = 0$, *i.e.* that $\operatorname{grad} G(p(t))$ is identically 0. By definition of $\operatorname{grad}(G)$, one has

$$\forall t \in]0,1[, \quad \frac{dG}{dt}(p(t)) = \left\langle \frac{dp}{dt} \mid \operatorname{grad} G(p(t)) \right\rangle. \tag{1}$$

Therefore $G(p(t))$ is a constant ν, different from 0 and ∞ by the hypothesis of the lemma. This contradicts the fact that G is semitame as ν belongs to the bifurcation set B. Then in fact $\mathbf{c} \neq 0$.

Replacing each term by its expansion in the equality $\operatorname{grad} G(p(t)) = \lambda(t)p(t)\overline{G(p(t))}$, we get

$$\mathbf{c}t^\gamma + \ldots = \lambda(t)(\mathbf{a}t^\alpha + \ldots)(\bar{b}t^\beta + \ldots).$$

Identifying the coefficients of lower degree, we obtain $\lambda(t) = \lambda_0 t^{\gamma-\alpha-\beta} + \ldots$, and $\mathbf{c} = \lambda_0 \mathbf{a}\bar{b}$. Then from (1) we obtain

$$\beta b t^{\beta-1} + \ldots = \alpha \|a\|^2 \bar{\lambda}_0 b t^{\alpha-1+\gamma} + \ldots$$

Assume that $\beta = 0$ then $\lim_{t\to 0} G(p(t)) = b \in \mathbb{C}^*$ (and $\alpha + \gamma > 0$ which implies $\|p(t)\| \cdot \|\operatorname{grad} G(p(t))\| \to 0$) and b belongs to the bifurcation set B. This contradicts the fact that G is semitame. Then in fact $\beta \neq 0$.

Therefore, $\beta = \alpha\|a\|^2\bar{\lambda}_0$ which proves that λ_0 is a non-zero real number. $\square$

Another modification of the proof of Milnor to apply the curve selection lemma is to transform all equalities involving meromorphic function into real analytic equalities. For instance, for $G = \frac{f}{g}$ let us consider the set $M(G)$ of all $z \in \mathbb{C}^n$ for which the vectors $\operatorname{grad} G(z)$ and z are linearly dependent, as in Lemma 4.3 of [11]. Then the equation $\operatorname{grad} G(z) = \lambda z$, where $z = (z_1, \ldots, z_n)$, is equivalent to the system of analytic equations:

$$f\frac{\partial g}{\partial z_i} - g\frac{\partial f}{\partial z_i} = \lambda z_i \bar{g}^2, \qquad i = 1, \ldots, n.$$

As in Milnor's proof these equations can be transformed into real analytic equations with real variables $(x_1, y_1, \ldots, x_n, y_n)$, where $x_i = \operatorname{Re}(z_i)$ and $y_i = \operatorname{Im}(z_i)$. Then the set $M(G)$ is a real analytic set.

In the next paragraphs, we restrict ourselves to the case $n = 2$.

The Milnor map and the local multilink. At first, let us recall the notion of fibration of a multilink in $\mathbb{S}^3$. For more details see [3].

Definition 2.8. A multilink $L = n_1K_1 \cup \ldots \cup n_\ell K_\ell$ in $\mathbb{S}^3$ is fibred if there exists a map $\Phi : \mathbb{S}^3 \setminus L \longrightarrow \mathbb{S}^1$ which satisfies the following two conditions:

(1) The map Φ is a C^∞ locally trivial fibration;
(2) For each $i = 1, \ldots, \ell$, let $m_i = \partial D_i$ be the boundary of a meridian disk of a small tubular neighbourhood of K_i oriented in such a way that $D_i \cdot K_i = +1$ in $\mathbb{S}^3$. The degree of the restriction of Φ to m_i equals n_i.

The following is obtained by examining the behaviour of the map $\pi \circ \frac{f/g}{|f/g|}$ near each component of the strict transform of $f \cdot g$, where $\pi : \Sigma \to \mathbb{C}^2$ denotes a resolution of the germ $f \cdot g$. For details see [14, Proposition 3.1] or [15, Lemma 5.1].

Proposition 2.9. *Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors. If the Milnor map*

$$\frac{f/g}{|f/g|} : \mathbb{S}^3_\varepsilon \setminus L_{f/g} \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration, then it is a fibration of the multilink $L_{f/g}$.

The bifurcation set B and the special fibres of the pencil. Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors. Let $V \subset \mathbb{C}$ be a Zariski open. Let us denote by $(f/g)^{-1}(t)$ the germ of plane curve at the origin of $\mathbb{C}^2$ with equation $f(x, y) - tg(x, y) = 0$. The pencil of curves $((f/g)^{-1}(t))_{t \in V}$ is *equisingular* if for all $t_1, t_2 \in V$, the curves $(f/g)^{-1}(t_1)$ and $(f/g)^{-1}(t_2)$ are equisingular in the sense of Zariski. There exists a maximal $V_{\max}$ with this property (see e.g. [8]).

Definition 2.10. The set $B' = \mathbb{P}^1 \setminus V_{\max}$ is called the set of the *special fibres* of the pencil of plane curves generated by f and g, or equivalently, the special fibres of the meromorphic function f/g.

By convention and to avoid discussions we set

$$\{0, \infty\} \subset B'.$$

The following result is a meromorphic and local version of parts of the well-known equivalence of the different definitions of a critical value at infinity for a polynomial map, see [2] for a survey and proofs, and also [6],[8], [13], [16].

Proposition 2.11. $(n = 2)$
For $c \notin \{0, \infty\}$, the following assertions are equivalent:

1. *$c \notin B$;*
2. *The topological type of the germ of curve $(f/g)^{-1}(t)$ is constant for all t near c;*
3. *c is a regular value of the map Φ obtain from the resolution of f/g at the origin;*
4. *$c \notin B'$.*

Corollary 2.12. $(n = 2)$

$$B = B'.$$

A condition for fibration in terms of bifurcation sets.

Proposition 2.13. *Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors. If the link $L_{f/g}$ of f/g is fibred then $B' = \{0, \infty\}$.*

Proof. The proof follows the one of [1, Theorem 2] excepted for the case where the link $L_{f/g}$ (or the underlying link if it has multiplicities) is the Hopf link.

We briefly recall the ideas from the proof. We suppose that $c \in B'$ with $c \notin \{0, \infty\}$, then

$$F = \left(\frac{f/g}{|f/g|}\right)^{-1}\left(-\frac{c}{|c|}\right) \cap \mathbb{S}^3_\varepsilon,$$

is a Seifert surface for the link $L_{f/g}$. According to Proposition 2.11 there exists a dicritical divisor D of the resolution of f/g at 0 that is of valency 3, that is to say it corresponds to a Seifert manifold in the minimal Waldhausen decomposition of $\mathbb{S}^3_\varepsilon \setminus L_{f/g}$. For ω sufficiently near c, there exists a connected component ℓ of the link $(f/g)^{-1}(\omega) \cap \mathbb{S}^3_\varepsilon$ corresponding to D. Clearly $\ell \cap F = \varnothing$. Then by the characterisation of fibred links in [3, Theorem 11.2] if $L_{f/g}$ is not the Hopf link then $L_{f/g}$ is not fibred.

For the Hopf link, up to an analytical change of coordinates we can suppose $f(x,y) = x^p$ and $g(x,y) = y^q$, that implies $B' = \{0, \infty\}$. □

Remark 2.14. There is an alternative proof using the results of [9] and [10] about another bifurcation set B'' defined in terms of the Jacobian curve of the morphism $(f,g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$: if the multilink $L_{f/g}$ is fibred, then [9, Theorem 1.1] implies that 1 is not a Jacobian quotient of the germ $(f,g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Then, by Remark 3 and Theorem 1 of [10], any $c \notin \{0, \infty\}$ is a generic value of the pencil generated by f and g.

Summary. Theorem 2.6, Corollary 2.12 and Proposition 2.13 lead to the following Theorem :

Theorem 2.15. *Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors. The following are equivalent:*

(1) *The meromorphic function f/g is semitame at the origin;*
(2) *Each $c \notin \{0, \infty\}$ is a generic value of the pencil of curves generated by f and g;*
(3) *The multilink $L_{f/g}$ is fibred.*

Moreover, if these conditions hold, then the Milnor map

$$\frac{f/g}{|f/g|} : \mathbb{S}^3_\varepsilon \setminus L_{f/g} \longrightarrow \mathbb{S}^1$$

is a fibration of the multilink $L_{f/g}$.

Proof. (1) ⇒ (3) is Theorem 2.6 and Proposition 2.9
(3) ⇒ (2) is Proposition 2.13
(3) ⇔ (1) is Proposition 2.12 □

Now, Theorem 2.15 and Theorem 2 of [15] can be summarized in the statement of Theorem 1.2 of the introduction.

3. Milnor map of a meromorphic function: the global case

We now produce a very similar description for singularities at infinity, which leads to a complete answer to the question of Hirasawa and

Rudolph. The statements and the proofs are directly adapted from that of Section 2.

Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ be two polynomials with no common factor. For short, we denote by f/g the meromorphic map $f/g : \mathbb{C}^2 \longrightarrow \mathbb{C}$ well-defined outside $I(f/g)$. Recall the $L_{f,\infty}$ and $L_{g,\infty}$ denotes the multilinks at infinity $f^{-1}(0) \cap \mathbb{S}_R^{2n-1}$ and $g^{-1}(0) \cap \mathbb{S}_R^{2n-1}$, $R \gg 1$, of f and g respectively.

Definition 3.1. The *multilink at infinity* $L_{f/g,\infty}$ of the meromorphic function f/g is defined by

$$L_{f/g,\infty} = L_{f,\infty} \cup -L_{g,\infty}$$

We will state a fibration theorem for $L_{f/g,\infty}$ under a semitame condition. Then for $n = 2$ we will state the reciprocal. We again consider the set $M(f/g)$ defined as in Section 2 (where U is now $\mathbb{C}^n$) and we define a bifurcation set $B_\infty \subset \mathbb{P}^1$ for the meromorphic function $G = f/g$ at infinity as follows.

Definition 3.2. The set B_∞ consists of all values $c \in \mathbb{P}^1$ such that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of $M(G)$ such that

$$\lim_{k \to \infty} \|x_k\| = +\infty \quad \text{and} \quad \lim_{k \to \infty} G(x_k) = c.$$

By convention, we set

$$\{0, \infty\} \subset B_\infty.$$

Definition 3.3. The meromorphic map f/g is *semitame at infinity* if

$$B_\infty = \{0, \infty\}.$$

As for polynomial maps at infinity ([12], [1]) we can state a Milnor fibration theorem which is the adaptation of Theorem 2.6 and Proposition 2.9 to the global case.

Theorem 3.4. *Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ be two polynomials with no common factor. If the meromorphic function is semitame at infinity, then there exists $R_0 \gg 1$ such that for each $R \geqslant R_0$, the Milnor map*

$$\frac{f/g}{|f/g|} : \mathbb{S}_R^{2n-1} \setminus L_{f/g,\infty} \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration. Moreover it is a fibration of the multilink at infinity $L_{f/g,\infty}$.

In the sequel, we restrict to the case $n = 2$. Let $f, g \in \mathbb{C}[x, y]$ be two polynomials with no common factor. We denote by $\tilde{f} \in \mathbb{C}[x, y, t]$ and $\tilde{g} \in \mathbb{C}[x, y, t]$ the homogenisations of f and g. The meromorphic map $\tilde{G} = [\frac{\tilde{f}}{\tilde{g}} : 1] : \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ may not be defined at some points on the line at infinity $H_\infty = \{t = 0\}$ of $\mathbb{P}^2$. In the case $\deg f = \deg g$, then the restriction $\tilde{G}_| : H_\infty \longrightarrow \mathbb{P}^1$ is a ramified covering.

Definition 3.5. A point of H_∞ where $\tilde{G}$ is not well-defined is an *indetermination point* of $\tilde{G}$, and a point of H_∞ where $\tilde{G}$ is well-defined but the restriction $\tilde{G}_| : H_\infty \to \mathbb{P}^1$ is ramified is a *ramification point* of $\tilde{G}$.

Let $\pi : \Sigma \longrightarrow \mathbb{P}^2$ be a resolution of the meromorphic function $\tilde{G}$, *i.e.* the composition of a finite sequence of blows-up of points starting with the blows-up of the indetermination and of the ramification points of $\tilde{G}$ such that there exists a map $\hat{G} : \Sigma \longrightarrow \mathbb{P}^2$ such that $\hat{G} = \tilde{G} \circ \pi$ which is well defined on $\pi^{-1}(H_\infty)$

The following is the analogous of Proposition 2.11.

Proposition 3.6. *For $n = 2$, $c \notin \{0, \infty\}$, the following assertions are equivalent:*

1. *$c \notin B_\infty$;*
2. *outside a large compact set of $\mathbb{C}^2$, the topological type of the curve ${f/g}^{-1}(s)$ is constant for all s near c;*
3. *c is a regular value of the map $\hat{G}$.*

Proposition 3.6 and the arguments of the proof of Proposition 2.13 (or [1], Theorem 2) lead to the following:

Proposition 3.7. *Let $f, g \in \mathbb{C}[x, y]$ be two polynomials with no common factor. If the multilink at infinity $L_{f/g,\infty}$ of f/g is fibred then $B_\infty = \{0, \infty\}$.*

These results enable one to answer positively to the question of M. Hirasawa and L. Rudolph [7]:

Theorem 3.8. *Let $f, g \in \mathbb{C}[x, y]$ be two polynomials with no common factor. The following conditions are equivalent:*

1. *The meromorphic map f/g is semitame at infinity;*
2. *The multilink at infinity $L_{f/g,\infty}$ is fibred.*

Moreover, if these conditions hold, then the Milnor map

$$\frac{f/g}{|f/g|} : \mathbb{S}^3_R \setminus L_{f/g,\infty} \longrightarrow \mathbb{S}^1$$

is a fibration of the multilink $L_{f/g,\infty}$.

Example 3.9. Let us take an example from [7] : $G = \frac{x^2-y^2}{x^2+y^2}$. Then the link at infinity is not fibred and we have $B_\infty = \{0, +1, -1, \infty\}$.

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Milnor fibrations of meromorphic functions

avec Anne Pichon et José Seade

MILNOR FIBRATIONS OF MEROMORPHIC FUNCTIONS

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Abstract. In analogy with the holomorphic case, we compare the topology of Milnor fibrations associated to a meromorphic germ f/g : the local Milnor fibrations given on Milnor tubes over punctured discs around the critical values of f/g, and the Milnor fibration on a sphere.

1. Introduction

The classical fibration theorem of Milnor in [8] says that every holomorphic map (germ) $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with a critical point at $0 \in \mathbb{C}^n$ has two naturally associated fibre bundles, and both of these are equivalent. The first is:

$$\phi = \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \longrightarrow \mathbb{S}^1 \tag{1}$$

where $\mathbb{S}_\varepsilon$ is a sufficiently small sphere around $0 \in \mathbb{C}^n$ and $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link of f at 0. The second fibration is:

$$f : \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1 \tag{2}$$

where $\mathbb{B}_\varepsilon$ is the closed ball in $\mathbb{C}^n$ with boundary $\mathbb{S}_\varepsilon$ and $\mathbb{D}_\delta$ is a disc around $0 \in \mathbb{C}$ which is sufficiently small with respect to ε.

The set $N(\varepsilon, \delta) = \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\delta)$ is usually called a local *Milnor tube* for f at 0, and it is diffeomorphic to $\mathbb{S}_\varepsilon$ minus an open regular neighbourhood T of K. (Thus, to get the equivalence of the two fibrations one has to "extend" the latter fibration to $T \setminus K$.) In fact, in order to have the second fibration one needs to know that every map-germ f as above has the so-called "Thom property", which was not known when Milnor wrote his book. What he proves is that the fibers in (1) are diffeomorphic to the intersection $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ for t close enough to 0. The statement that (2) is a fibre bundle was proved later in [6] by Lê Dũng Tráng in the more general setting of holomorphic maps defined on arbitrary complex analytic spaces, and we call it the *Milnor-Lê* fibration of f. Once we know that (2) is a fibre bundle, the arguments of [8, Chapter 5] show this is equivalent to the *Milnor* fibration (1).

The literature about these fibrations is vast, and so are their generalizations to various settings, including real analytic map-germs and meromorphic maps, and that is the starting point of this article.

Let U be an open neighbourhood of 0 in $\mathbb{C}^n$ and let $f, g : U \longrightarrow \mathbb{C}$ be two holomorphic functions without common factors such that $f(0) = g(0) = 0$.

Let us consider the meromorphic function $F = f/g : U \to \mathbb{C}P^1$ defined by $(f/g)(x) = [f(x)/g(x)]$. Notice that f/g is not defined on the whole U ; its indetermination locus is

$$I = \left\{ z \in U \mid f(x) = 0 \text{ and } g(x) = 0 \right\} .$$

In particular, the fibers of $F = f/g$ do not contain any point of I : for each $c \in \mathbb{C}$, the fiber $F^{-1}(c)$ is the set

$$F^{-1}(c) = \left\{ x \in U \mid f(x) - cg(x) = 0 \right\} \setminus I .$$

In a series of articles, S. M. Gusein-Zade, I. Luengo and A. Melle-Hernández, and later D. Siersma and M. Tibăr, studied local Milnor fibrations of the type (2) associated to every critical value of the meromorphic map $F = f/g$. See for instance [4, 5], or Tibar's book [14] and the references in it. Of course the "Milnor tubes" $\mathbb{B}_\varepsilon \cap F^{-1}(\partial \mathbb{D}_\delta)$ in this case are not actual tubes in general, since they may contain $0 \in U$ in their closure. These are in fact "pinched tubes".

It is thus natural to ask whether one has for meromorphic map-germs fibrations of Milnor type (1), and if so, how these are related to those of the Milnor-Lê type (2) studied (for instance) in [4, 5, 14]. The first of these questions was addressed in [12, 1, 13] from two different viewpoints, while the answer to the second question is the bulk of this article.

In fact, it is proved in [1] that if the meromorphic germ $F = f/g$ is semitame (see the definition in Section 2), then

$$\frac{F}{|F|} = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1 \tag{3}$$

is a fiber bundle, where $L_f = \{f = 0\} \cap \mathbb{S}_\varepsilon$ and $L_g = \{g = 0\} \cap \mathbb{S}_\varepsilon$ are the oriented links of f and g. Notice that away from the link $L_f \cup L_g$ one has an equality of maps:

$$\frac{f/g}{|f/g|} = \frac{f\bar{g}}{|f\bar{g}|} ,$$

where $\bar{g}$ denotes complex conjugation. It is proved in [13] that if the real analytic map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$ and satisfies the Thom property, then the Milnor-Lê fibration of $f\bar{g}$,

$$N(\varepsilon,\delta) := [\mathbb{B}_\varepsilon \cap (f\bar{g})^{-1}(\partial\mathbb{D}_\delta)] \xrightarrow{f\bar{g}} \partial\mathbb{D}_\delta \cong \mathbb{S}^1\,, \tag{4}$$

is equivalent to the Milnor fibration (3) of f/g when this map is semitame. That is, the fibration (4) on the Milnor tube $N(\varepsilon,\delta)$ of $f\bar{g}$ is equivalent to the Milnor fibration (3) of the meromorphic germ f/g.

In this article we complete the picture by comparing the local fibrations of Milnor-Lê type of a meromorphic germ f/g studied by Gusein-Zade *et al*, with the Milnor fibration (3). We prove that if the germ f/g is semitame and (i)-tame (see Sections 2 and 3), then the global Milnor fibration (3) for f/g is obtained from the local Milnor fibrations of f at 0 and ∞ by a gluing process that is, fiberwise, reminiscent of the classical connected sum of manifolds (see Theorem 9, and its corollaries, in Section 5).

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2. Semitameness and the global Milnor fibration of F

Adapting Milnor's definition [8], we define the gradient of $F = f/g$ at a point $x \in U \setminus I$ by :

$$\operatorname{grad}(f/g) = \left(\overline{\frac{\partial(f/g)}{\partial x_1}}, \dots, \overline{\frac{\partial(f/g)}{\partial x_n}} \right).$$

The following definitions were introduced in [1] following ideas of [9]. We consider the set

$$M(F) = \left\{ x \in U \setminus I \mid \exists \lambda \in \mathbb{C}, \operatorname{grad}(f/g)(x) = \lambda x \right\}$$

consisting of the points of non-transversality between the fibres of f/g and the spheres $\mathbb{S}_r$ centered at the origin of $\mathbb{C}^n$.

Definition 1. A value $c \in \mathbb{C}P^1$ is called *atypical* if there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $M(F)$ such that

$$\lim_{k\to\infty} x_k = 0 \quad \text{and} \quad \lim_{k\to\infty} F(x_k) = c\,.$$

Otherwise it is called typical. The finite set B of the atypical values is called the *bifurcation set* of the meromorphic function f/g.

Let $L_f = \{f = 0\} \cap \mathbb{S}_\varepsilon$ and $L_g = \{g = 0\} \cap \mathbb{S}_\varepsilon$ be the oriented links of f and g.

Proposition 2. *Let W be an open set in $\mathbb{C}P^1$ such that $W \cap B = \emptyset$. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \leqslant \varepsilon_0$, the map*

$$\Phi_W = \frac{f/g}{|f/g|} : \big(\mathbb{S}_\varepsilon \setminus (L_f \cup L_g)\big) \cap F^{-1}(W) \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration.

The proof is that of [1, Theorem 2.6]; it follows Milnor's proof [8, Chapter 4] with minor modifications. See also [9]. The main modification of Milnor's proof concerns Lemma 4.4 of [8], for which an adapted formulation and a detailed proof is given in [1, Lemma 2.7].

Definition 3. The meromorphic function f/g is *semitame* at 0 if $B \subset \{0, \infty\}$.

Proposition 2 is a more general statement than [1, Theorem 2.6]. When F is semitame, the following is obtained by applying Proposition 2 to $W = \mathbb{C}P^1 \setminus \{0, \infty\}$:

Corollary 4. *(*[1, Theorem 2.6]*) If F is semitame, then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \leqslant \varepsilon_0$, the map*

$$\Phi_F = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon^{2n-1} \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration.

Definition 5. When F is semitame, we call Φ_F *the global Milnor fibration* of the meromorphic germ F.

It is shown in [13] that Φ is a fibration of the multilink $L_f \cup -L_g$, where $-L_g$ means L_g with the opposite orientation.

For our purpose, it will be necessary to consider the restriction $\check{\Phi}_F$ of Φ_F to $\big(\mathbb{S}_\varepsilon^{2n-1} \setminus (L_f \cup L_g)\big) \setminus F^{-1}(\mathbb{D}_\delta(0) \cup \mathbb{D}_R(0))$ where $\delta \ll 1$ and $1 \ll R$.

Definition 6. We denote by $\check{\mathcal{M}}_F$ the fibre of $\check{\Phi}_F$ and we call it the *truncated global Milnor fibre* of F.

3. Tameness near the indetermination points

In this section we introduce a technical condition on f/g: the (i)-tameness ((i) for "indetermination") which enables us to control the behaviour of f/g in a neighbourhood of its indermination points when

$n \geqslant 3$. This condition will appear as an essential hypothesis for our main Theorem 9. Note that this section only concerns the case $n \geqslant 3$.

Let us fix $r > 0$ and let us consider some sufficiently small constants $0 < \varepsilon' \ll \delta \ll \varepsilon \ll 1$. These constant will be defined more precisely in the proof of Theorem 9.

Let $X = F^{-1}\left(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)\right) \cap \left(\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}\right)$.

For $\eta > 0$, we consider the neighbourhood of I defined by:

$$N_\eta = \left\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 \leqslant \eta^2\right\},$$

and its boundary,

$$\partial N_\eta = \left\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 = \eta^2\right\}.$$

The proof of Theorem 9 is based on the existence of a vector field v on X which satisfies for all sufficiently small η, $0 < \eta \ll \varepsilon'$ the following properties:

(1): The argument of f is constant along the integral curves of v.
(2): The norm of z is strictly increasing along the integral curves of v.
(3): For all $z \in N_\eta$, the integral curve passing through z is contained in the tube $\partial N_{\eta'}$ where ${\eta'}^2 = |f(z)|^2 + |g(z)|^2$.

In this paper, we use two different inner products on $\mathbb{C}^n$:

(1) The usual hermitian form $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined for $z = (z_1, \ldots, z_n), z' = (z'_1, \ldots, z'_n) \in \mathbb{C}^n$ by :

$$\langle z, z' \rangle = \sum_{k=1}^{n} z_k \bar{z'_k}$$

(2) The usual inner product $\langle\ ,\ \rangle_{\mathbb{R}} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ on $\mathbb{R}^{2n}$:

$$\langle z, z' \rangle_{\mathbb{R}} = \sum_{k=1}^{n} (x_k x'_k + y_k y'_k),$$

where $\forall k, z_k = x_k + iy_k$ and $z'_k = x'_k + iy'_k$.

Notice that for $z, z' \in \mathbb{C}^n$,

$$\langle z, z' \rangle = \langle z, z' \rangle_{\mathbb{R}} + i\langle z, iz' \rangle_{\mathbb{R}}$$

As we will show in the proof of Theorem 9, the semitameness of f/g guarantees the existence of a vector field v on X such that:

(i) For all $z \in X, \langle v(z), \operatorname{grad} \log F(z) \rangle = +1$.
(ii) For all $z \in X \setminus M(F)$, $\langle v(z), z \rangle > 0$.
(iii) For all $z \in U$, $Re\langle v(z), z \rangle > 0$".

So that conditions (1) and (2) are satisfied. We now introduce an additional hypothesis which will ensure that (3) is also satisfied, *i.e.* that v is such that :

(iv) For all $z \in X \cap N_\eta \setminus I$ one has $v(z) \in T_z \partial N_{\eta'}$, where $\eta'^2 = |f(z)|^2 + |g(z)|^2$.

As shown in the proof of the Theorem 9, semitameness is sufficient to define such a v in a neighbourhood of $M(F) \cap N_\eta$. Now, let $z \in N_\eta \setminus M(F)$.

We set $\gamma(z) = |f(z)|^2 + |g(z)|^2$ so that

$$T_z \partial N_{\eta'} = \left\{ v \in \mathbb{R}^{2n} \mid \langle v, \operatorname{grad}_{\mathbb{R}} \gamma(z) \rangle_{\mathbb{R}} = 0 \right\}.$$

Then a vector $v \in \mathbb{R}^{2n}$ satisfies (i), (ii) and (iv) if and only if

$$\langle v, \operatorname{grad} \log F(z) \rangle = +1 \ , \langle v, z \rangle > 0 \ \text{ and } \ \langle v, \operatorname{grad}_{\mathbb{R}} \gamma(z) \rangle_{\mathbb{R}} = 0 \, .$$

Such a v exists if and only if $\operatorname{grad}_{\mathbb{R}} \gamma(z)$ does not belong to the $\mathbb{C}$-vector space generated by z and $\operatorname{grad} \log F(z)$, or equivalently by z and $\operatorname{grad} F(z)$. This makes natural the following definition. We set:

$$N(F) = \left\{ z \in U \setminus I \;\middle|\; \exists \lambda, \mu \in \mathbb{C}, \ \operatorname{grad}_{\mathbb{R}} \gamma(z) = \lambda z + \mu \operatorname{grad} F(z) \right\}.$$

Definition 7. Let $n \geqslant 3$. We say that $f/g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is (i)-tame if there exist sufficiently small constants $0 < \eta \ll \varepsilon' \ll \delta \ll \varepsilon \ll 1$ such that

$$\big(N(F) \cap N_\eta \cap X \big) \subset \big(M(F) \cap N_\eta \cap X \big) \, .$$

When $n = 2$, we define the (i)-tameness as an empty condition.

Notice that (i)-tameness is a generic property in the following sense. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ without common branches. Then the set of indetermination points $I = \{ z \in \mathbb{C}^n \mid f(z) = g(z) = 0 \}$ has complex dimension $n-2$. Moreover, $N(F) \cup M(F)$ is included in the set

$$P(F) = \{ z \in \mathbb{C}^n \mid \operatorname{rank} A(z) < 3 \} \, ,$$

where $A(z)$ is the matrix

$$\begin{pmatrix} \overline{\frac{\partial f}{\partial z_1}} f + g \overline{\frac{\partial g}{\partial z_1}} & \overline{\frac{\partial f}{\partial z_2}} f + g \overline{\frac{\partial g}{\partial z_2}} & \ldots & \overline{\frac{\partial f}{\partial z_n}} f + g \overline{\frac{\partial g}{\partial z_n}} \\ z_1 & z_2 & \ldots & z_n \\ \overline{\frac{\partial f}{\partial z_1} g - \frac{\partial g}{\partial z_1} f} & \overline{\frac{\partial f}{\partial z_2} g - \frac{\partial g}{\partial z_2} f} & \ldots & \overline{\frac{\partial f}{\partial z_n} g - \frac{\partial g}{\partial z_n} f} \end{pmatrix}$$

Then $P(F)$ is generically a real analytic submanifold of $\mathbb{C}^n$ with real codimension $2n-4$. Then generically, the two germs of analytic submanifolds $(I,0)$ and $(N(F)\cup M(F),0)$ intersect only at 0. Therefore, when the constants $0<\eta\ll\varepsilon'\ll\delta\ll\varepsilon\ll 1$ are sufficiently small, we obtain $P(F)\cap N_\eta\cap X=\emptyset$, and then, f/g is (i)-tame.

Example 1. It may happen that f/g is (i)-tame even if I is contained in $N(F)\cup M(F)$. For example, let $f,g:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ be defined by $f(x,y,z)=x^p$ and $g(x,y,z)=y^q$. Then the set of indetermination points of f/g is the z-axis, and the set $P(f/g)$ has equation $\det A(x,y,z)=0$, *i.e.* :

$$zx^{p-1}y^{q-1}(|x|^{2p}+|y|^{2q})=0\,.$$

Then $N(f/g)$ is included in the plane $\{z=0\}$ and f/g is (i)-tame, whereas $I\subset P(f/g)$. Hence f/g is also semitame.

Example 2. Let $f=f(x,y)$ and $g=g(x,y)$ be considered as germs from $(\mathbb{C}^3,0)$ to $(\mathbb{C},0)$. Then the set of indetermination points of f/g is again the z-axis, and the set $P(f/g)$ has equation

$$z\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right)\left(|f|^2+|g|^2\right)=0\,.$$

Therefore f/g is (i)-tame if and only if the jacobian curve $\{\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}=0\}$ of the germ $(f,g):(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ is included in the curve $\{fg=0\}$.

On the other hand, it is easy to obtain examples with f/g semitame. For instance, with f,g as above, if we regard f/g as a map-germ at $0\in\mathbb{C}^3$, then this is semitame if f/g is semitame as a germ from $(\mathbb{C}^2,0)$ into $(\mathbb{C},0)$, since a sequence of bad points (z_k) for f/g would project on the plane $z=0$ to a sequence of bad points for $f/g:(\mathbb{C}^2,0)\to(\mathbb{C},0)$. Now, it is easy to check whether $f/g:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ is semitame by using the characterization of semitameness given in [1, Theorem 1] when $n=2:$ f/g is semitame if and only if the multilink $L_f\cup -L_g$ is fibered. This latter condition is easily checked by computing a resolution graph of the meromorphic function f/g: the multilink $L_f\cup -L_g$ is fibered if and only if the multiplicities of f and g are different on each rupture component of the exceptional divisor of f/g.

Example 3. Let $f(x,y)=x^3+y^2$ and $g(x,y)=x^2+y^3$. Then f/g is semitame, as can be seen on the resolution graph of f/g represented on Figure 1. The number between parenthesis on each vertex is the

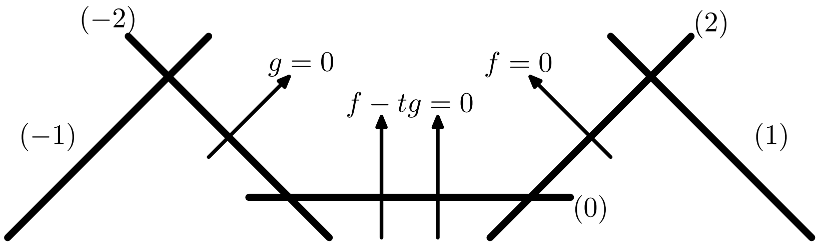

FIGURE 1. Resolution graph of $x^3 + y^2/x^2 + y^3$

difference $m_f - m_g$ where m_f (respectively m_g) is the multiplicity of f along the corresponding component of the exceptional divisor.

But f/g, seen as a map in variables (x, y, z) is not (i)-tame, because the germ of Jacobian curve $(J, 0)$ of (f, g) has equation $xy = 0$ and $N(f/g) = J \setminus I$.

4. The local Milnor fibrations of F

The local Milnor fibers of a meromorphic function F were defined in [4] as follows. Let us fix $c \in \mathbb{C}P^1$. There exists $\varepsilon_0 > 0$ such that for any ε, $0 < \varepsilon \leqslant \varepsilon_0$, the restriction $F_| : \mathbb{B}_\varepsilon \setminus (f = 0) \cup (g = 0) \to \mathbb{C}P^1$ defines a C^∞ locally trivial fibration over a punctured neighbourhood Δ_c of the point c in $\mathbb{C}P^1$.

Definition 8. The fiber $\mathcal{M}_F^c = F^{-1}(c') \cap \mathbb{B}_\varepsilon, c' \in \Delta_c$ of this fibration is called the *c-Milnor fiber* of F.

Notice that $\mathcal{M}_F^c$ is a non-compact n-dimensional manifold with boundary.

Let δ, $0 < \delta \ll \varepsilon$, be such that $\mathbb{D}_\delta(c) \subset \Delta_c$. We call the restriction

$$\phi_c = F_| : F^{-1}(\mathbb{S}^1_\delta(c)) \cap B_\varepsilon \longrightarrow \mathbb{S}^1_\delta(c)$$

the *c-local Milnor fibration* of the meromorphic map F.

According to [4, Lemma 1], the diffeomorphism class of the non-compact n-complex manifold $\mathcal{M}_F^c$ does not depend on ε, and the isomorphism class of the fibration ϕ_c does not depend on ε and δ. As shown in [14], this is in fact an immediate consequence of Lê's fibration theorem in [6] applied to the pencil $\{f - tg = 0\}$.

For our purpose, it will be necessary to consider the restriction of ϕ_c to the complement in $\mathbb{B}_\varepsilon$ of a small ball $\mathbb{B}_{\varepsilon'}$, $0 \ll \varepsilon' \ll \delta$, defined as follows. Since $\Delta_c \cap B = \emptyset$, there exists ε', $0 < \varepsilon' \ll \delta \ll \varepsilon$, such that $M(F) \cap F^{-1}(\mathbb{S}^1_\delta(c)) \cap \mathbb{B}_{\varepsilon'} = \emptyset$. For such an ε', we consider the restriction of the c-local Milnor fibration

$$\check{\phi}_c = F_{|} : F^{-1}(\mathbb{S}^1_\delta(c)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(c)\,.$$

And we denote by $\check{\mathcal{M}}^c_F = \mathcal{M}^c_F \setminus \mathring{\mathbb{B}}_{\varepsilon'}$ the fiber of $\check{\phi}_c$

Again, the diffeomorphism class of $\check{\mathcal{M}}^c_F$ and the isomorphism class of $\check{\phi}_c$ do not depend on ε, δ and ε'.

5. The results

Theorem 9. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common branches such that $F = f/g$ is (i)-tame. Then for all $r \in \mathbb{C}P^1 \setminus \{0, \infty\}$ such that $B \cap \mathbb{D}_r(0) = \{0\}$, there exist $\varepsilon, \varepsilon'$ and δ, $0 < \varepsilon' \ll \delta \ll \varepsilon \ll 1$, such that the restricted 0-local Milnor fibration*

$$\check{\phi}_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0) \tag{5}$$

is diffeomorphic to the fibration

$$\Phi_W : \big(\mathbb{S}_\varepsilon \setminus (L_f \cup L_g)\big) \cap F^{-1}(W) \longrightarrow \mathbb{S}^1. \tag{6}$$

where $W = \mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)$.

Remember that $\check{\phi}_0$ is a restriction of F and Φ_W is a restriction of $\frac{F}{|F|}$.

Corollary 10. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common branches such that $F = f/g$ is semitame and (i)-tame. For $\delta \ll 1$ and $R \gg 1$ one has:*

- **a)** *The truncated global Milnor fiber $\check{\mathcal{M}}_F = \check{\Phi}_F^{-1}(1)$ is diffeomorphic to the union of the two restricted local Milnor fibers $\check{\mathcal{M}}^0_F = \check{\phi}_0^{-1}(\delta)$ and $\check{\mathcal{M}}^\infty_F = \check{\phi}_\infty^{-1}(R)$ glued along their boundary components $\partial_0 = \check{\phi}_0^{-1}(\delta) \cap \mathbb{S}_{\varepsilon'}$ and $\partial_\infty = \check{\phi}_\infty^{-1}(R) \cap \mathbb{S}_{\varepsilon'}$*

$$\check{\mathcal{M}}_F \simeq \check{\mathcal{M}}^0_F \cup_\partial \check{\mathcal{M}}^\infty_F\,.$$

- **b)** *The Euler characteristics verify:*

$$\chi\left(\check{\mathcal{M}}_F\right) = \chi\left(\check{\mathcal{M}}^0_F\right) + \chi\left(\check{\mathcal{M}}^\infty_F\right).$$

  and

$$\chi\left(\mathcal{M}_F\right) = \chi\left(\mathcal{M}^0_F\right) + \chi\left(\mathcal{M}^\infty_F\right).$$

- **c)** *The monodromies $\check{h}_0 : \check{\mathcal{M}}^0_F \to \check{\mathcal{M}}^0_F$ and $\check{h}_\infty : \check{\mathcal{M}}^\infty_F \to \check{\mathcal{M}}^\infty_F$ of the fibrations $\check{\phi}_0$ and $\check{\phi}_\infty$ are the restrictions of the monodromy $\check{h} : \check{\mathcal{M}}_F \to \check{\mathcal{M}}_F$ of the fibration $\check{\Phi}_F$.*

Proof of the Corollary. We apply Theorem 9 twice with $r = 1$. The first time as stated, the second time around ∞, or in other words, around 0 for g/f. The proof of Theorem 9 furnishes:

- a diffeomorphism Θ_0 from
$$\check{\phi}_0^{-1}(\delta) = F^{-1}(\delta) \cap \big(\mathbb{B}_\varepsilon \setminus \mathbb{B}_{\varepsilon'}\big)$$
to
$$\frac{F}{|F|}^{-1}(1) \cap \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_1(0) \setminus \mathring{\mathbb{D}}_\delta(0)),$$
such that $\Theta_0^{-1}(\mathbb{D}_1(0)) = \partial_0$,
- and a diffeomorphism Θ_∞ from
$$\big(\check{\phi}_0\big)^{-1}(\delta) = F^{-1}(R) \cap \big(\mathbb{B}_\varepsilon \setminus \mathbb{B}_{\varepsilon'}\big)$$
to
$$\frac{F}{|F|}^{-1}(1) \cap \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_R(0) \setminus \mathring{\mathbb{D}}_1(0)),$$
such that $\Theta_\infty^{-1}(\mathbb{D}_1(0)) = \partial_\infty$.

The intersection of the images of Θ_0 and Θ_1 is exactly
$$\Theta_0(\partial_0) = \Theta_\infty(\partial_\infty) = F^{-1}(1) \cap \mathbb{S}_\varepsilon$$

Then $\check{\mathcal{M}}_F = \check{\Phi}_F^{-1}(1)$ is diffeomorphic to the union of $\check{\phi}_0^{-1}(\delta)$ and of $\check{\phi}_\infty^{-1}(R)$ glued along their boundary components ∂_0 and ∂_∞. This proves statement a).

The Euler characteristic verifies $\chi(A\cup B) = \chi(A)+\chi(B)-\chi(A\cap B)$. As the intersection of the images of Θ_0 and Θ_1 is a closed oriented manifold of odd dimension, then its Euler characteristic is 0. This proves the first equation in statement b). For the second equation, notice $\mathcal{M}_F^0$ (respectively $\mathcal{M}_F^\infty$) retracts by deformation to $\check{\mathcal{M}}_F^0$ (respectively $\check{\mathcal{M}}_F^\infty$), and $\mathcal{M}_F$ retracts by deformation to $\check{\mathcal{M}}_F$, proving b).

The statement c) follows from a) and Theorem 9. □

Corollary 11. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common branches such that $F = f/g$ is semitame and (i)-tame. If f, g have an isolated singularity at 0, then*
$$\chi\left(\mathcal{M}_F\right) = (-1)^{n-1}\big(\mu(f,0) + \mu(g,0) - 2\mu(f+tg,0)\big).$$
Where t is a generic value (i.e., $t \neq 0, \infty$) and μ is the Milnor number.

Proof. According to [4, Theorem 2],
$$\chi\left(\mathcal{M}_F^0\right) = (-1)^{n-1}\big(\mu(f,0) - \mu(f+tg,0)\big)$$

and

$$\chi\left(\mathcal{M}_F^{\infty}\right) = (-1)^{n-1}\big(\mu(g,0) - \mu(f+tg,0)\big).$$

□

Corollary 12. *If $n = 2$, then the manifold $\mathcal{M}_F^0$ (respectively $\mathcal{M}_F^{\infty}$) has the homotopy type of a bouquet of circles. If we denote by λ_0 (respectively λ_∞) the number of circles in this bouquet. Then $\mathcal{M}_F$ is a bouquet of $\lambda_0 + \lambda_\infty + 1$ circles.*

6. Preliminary lemmas

The following lemmas are easily obtained by adapting the proofs of Lemmas 4.3 and 4.4 in [8] as already performed in [9] and [1] in close situations.

Lemma 13. *Assume that the meromorphic germ $F = f/g$ is semitame at the origin. Let $p : [0,1] \longrightarrow \mathbb{C}^n$ be a real analytic path with $p(0) = 0$ such that for all $t > 0$, $F(p(t)) \notin \{0, \infty\}$ and such that the vector* $\operatorname{grad}\log F(p(t))$ *is a complex multiple $\lambda(t)p(t)$ of $p(t)$. Then the argument of the complex number $\lambda(t)$ tends to 0 or π as $t \to 0$.*

Proof. Adapting [8, Lemma 4.4]. See also [9, Lemma 3] and [1, Lemma 2.7]. □

Lemma 14. *Let F be semitame. Then there exists $0 < \varepsilon \ll 1$ such that for all $z \in B_\varepsilon \setminus (F^{-1}(0) \cup F^{-1}(\infty))$ the two vectors z and* $\operatorname{grad}\log F(z)$ *are either linearly independent over $\mathbb{C}$ or* $\operatorname{grad}\log F(z) = \lambda z$ *with $|\arg(\lambda)| \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$.*

Proof. Using Lemma 13. See [8, Lemma 4.3] and [9, Lemma 4]. □

Lemma 15. *Let D', D'' be two 2-discs centered at 0 with $D' \subset D''$ and $D' \neq D''$. For $0 < \varepsilon \ll 1$, if $z \in \mathbb{S}_\varepsilon \setminus (F^{-1}(0) \cup F^{-1}(\infty))$ is such that* $\operatorname{grad}\log F(z) = \lambda z$, *($\lambda \in \mathbb{C}$) then*

$$F(z) \in D' \qquad \text{or} \qquad F(z) \notin D''.$$

Moreover in the first case $\arg(+\lambda) \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$ *and in the second case* $\arg(-\lambda) \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$.

Proof. Using Lemma 13 and Lemma 14. See [9, Lemma 8]. □

7. Proof of the theorem

First step: definition of the constants.

(1) Let $0 < r < \infty$ be such that $B \cap \mathbb{D}_r(0) = \{0\}$, where B is the bifurcation set of the meromorphic function f/g.

(2) As $\mathbb{S}^1_r(0)$ is compact and $\mathbb{S}^1_r(0) \cap B = \emptyset$, one can choose $0 < \varepsilon \ll 1$ such that:
  a) $\mathbb{B}_\varepsilon$ is a Milnor ball for $F^{-1}(0), F^{-1}(\infty)$, the indetermination set I and for all $F^{-1}(z), z \in \mathbb{S}^1_r(0)$;
  b) ε satisfies the conclusion of Lemma 15 for $D' = \mathbb{D}_{r/4}(0)$ and $D'' = \mathbb{D}_{r+1}(0)$.

(3) Let us choose δ, $0 < \delta \ll \varepsilon$, such that:

$$\phi_0 = F_| : F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(0)$$

is the 0-Milnor fibration of the meromorphic map F.

(4) Last, let us choose $\varepsilon'_0, 0 < \varepsilon'_0 \ll \delta$ such that

$$M(F) \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)) \cap \mathbb{B}_{\varepsilon'_0} = \emptyset,$$

and let us set $\varepsilon' = \varepsilon'_0/2$. In particular, one obtains the restricted 0-local Milnor fibration

$$\check{\phi}_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0)\,.$$

Let $\psi : U \to \mathbb{R}^2$ be defined by $\psi(z) = (\log|f(z)|, ||z||^2)$. Notice that Conditions 2.b) and 4) imply that

$$\big([\log\delta, \log r] \times [0, {\varepsilon'_0}^2] \cup [\log(r/2), \log r] \times [0, \varepsilon^2]\big) \cap \psi(M(F)) = \emptyset\,.$$

Second step : construction of a vector field.

Let us consider the set

$$X = F^{-1}\Big(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)\Big) \cap \big(\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}\big).$$

Let us fix $\rho \in]r/2, r[$ and $\varepsilon'' \in]\varepsilon', \varepsilon_0[$ and let us consider the two real numbers $0 < b_1 < b_2$ defined by:

$$b_1 = \frac{\varepsilon''^2 - \varepsilon'^2}{2(\log\rho - \log\delta)} \qquad \text{and} \qquad b_2 = \frac{\varepsilon_0^2 - \varepsilon''^2}{2(\log r - \log\rho)}\,.$$

Let us fix $\xi, 0 < \xi \ll \varepsilon''$ and an increasing C^∞ map $b : [0, \infty[\to [0, \infty[$ such that $\forall x \leqslant \varepsilon" - \xi, b(x) = b_1$ and $\forall x \geqslant \varepsilon" + \xi, b(x) = b_2$.

For $\eta > 0$, we consider the neighbourhood of I defined by:

$$N_\eta = \big\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 \leqslant \eta^2\big\},$$

and its boundary,

$$\partial N_\eta = \big\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 = \eta^2\big\}.$$

Let us fix η, $0 < \eta \ll \varepsilon'$, such that η satisfies the (i)-tameness condition :

$$\big(N(F) \cap N_\eta \cap X\big) \subset \big(M(F) \cap N_\eta \cap X\big)$$

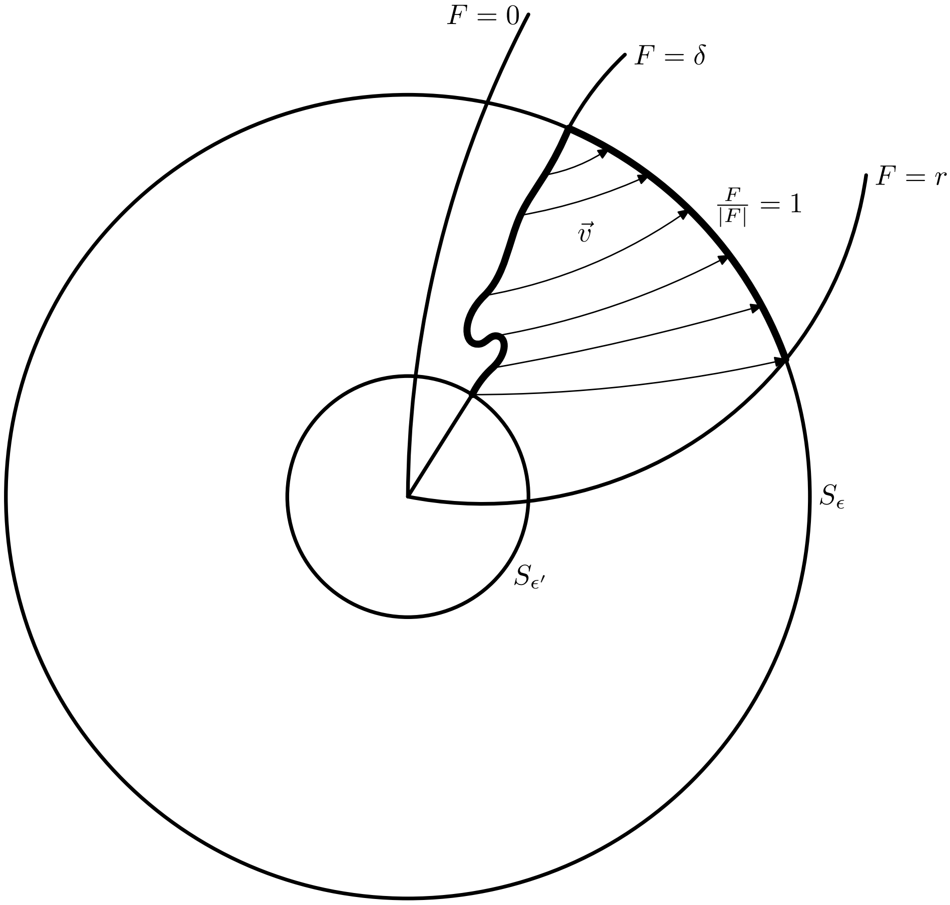

FIGURE 2. Vector field

Lemma 16. *There exists an open neighbourhood Ω of the set $M(F)$ in X, two real numbers α and β, $0 < \alpha < \beta$ and a differentiable vector field v on X such that:*

(i) *For all $z \in X, \langle v(z), \operatorname{grad} \log F(z) \rangle = +1$,*
(ii) *For all $z \in X \setminus M(F), \langle v(z), z \rangle = b(|z|)$*
(iii) *For all $z \in \Omega, Re\langle v(z), z \rangle \in [\alpha, \beta]$;*
(iv) *For all $z \in X \cap N_\eta$, $v(z) \in T_z \partial N_{\eta'}$ where ${\eta'}^2 = |f(z)|^2 + |g(z)|^2$.*

Proof. Let $\mu, 0 < \mu \ll \eta$, be such that $\eta + \mu$ again satisfies the (i)-tameness condition :

$$\big(N(F) \cap N_{\eta+\mu} \cap X\big) \subset \big(M(F) \cap N_{\eta+\mu} \cap X\big).$$

Let us denote by V the interior of $N_{\eta+\mu}$ in X, *i.e.*,

$$V = \{z \in X \mid 0 \leqslant |f(z)|^2 + |g(z)|^2 < (\eta + \mu)^2\},$$

and let us consider the four following open sets of X (the neighbourhood Ω of $M(F)$ will be defined later) :

$$U_1 = X \setminus (N_\eta \cup M(F)), \qquad U_2 = \Omega \setminus N_\eta$$
$$U_3 = V \cap \Omega, \qquad U_4 = V \setminus M(F)\,.$$

One has : $X = U_1 \cup U_2 \cup U_3 \cup U_4$. The vector field v will be obtained by constructing a vector field v_i on each U_i and by defining globally v by a partition of unity.

At first, let us define v on $X \setminus N_\eta = U_1 \cup U_2$. For a point $z \in U_1$, we define v_1 by using the classical construction of Milnor: for such a point the vectors z and $\operatorname{grad}\log F(z)$ are linearly independent over $\mathbb{C}$. Thus there exists $v_1(z)$ verifying (i) and (ii).

For each $z \in X$, let us consider the vector

$$u(z) = \frac{\operatorname{grad}\log F(z)}{||\operatorname{grad}\log F(z)||^2}\,.$$

Let $z \in M(F) \cap X$. There exists $\lambda \in \mathbb{C}$ such that $\operatorname{grad}\log F(z) = \lambda z$. Then $\langle u(z), \operatorname{grad}\log F(z)\rangle = +1$ and

$$\operatorname{Re}\langle u(z), z\rangle = \operatorname{Re}\left(\frac{\lambda}{|\lambda|^2}\right)\,.$$

Notice that $M(F) \cap \mathbb{B}_\varepsilon = \{z \in \mathbb{B}_\varepsilon \mid \exists \lambda \in \mathbb{C}, \operatorname{grad} F(z) = \lambda z\}$ is compact. Then $M(F) \cap X$ is a compact set, and there exist c_1, c_2, $0 < c_1 < c_2$ such that for all $z \in M(F) \cap X$ one has $c_1 < |\lambda| < c_2$ where λ is the complex number such that $\operatorname{grad} F(z) = \lambda z$. Moreover, Condition 2.b) below implies that $\arg(+\lambda) \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$. Then there exists $c_1', c_2' > 0$ such that for all $z \in M(F) \cap X$, $c_1' < \operatorname{Re}\langle u(z), z\rangle < c_2'$.

Let us choose ν such that $0 < \nu \ll c_1'$ and let us set $\alpha = c_1' - \nu$ and $\beta = c_2' + \nu$. There exists an open neighbourhood Ω of $M(F)$ in X such that for all $z \in \Omega$, $\alpha < \operatorname{Re}\langle u(z), z\rangle < \beta$. Then for each $z \in U_2 = \Omega \setminus N_\eta$, we set $v_2(z) = u(z)$.

We now define v on $V = U_3 \cup U_4$, *i.e.* near the indetermination set I. A picture of the local situation near I is represented on Figure 3. For a point z in V, we set

$$\eta' = \sqrt{|f(z)|^2 + |g(z)|^2}\,.$$

Let $T = T(z)$ be the space tangent to $\partial N_{\eta'}$ at z. We will construct a vector field v on V satisfying the three conditions (i), (ii) and (iii) and such that $v(z) \in T$.

If $z \in U_3 = V \cap \Omega$, let us again consider the vector $u(z)$. If $u(z) \in T$, then we set $v_3(z) = u(z)$. If $u(z) \notin T$, let

$$Q = \big(u(z)\big)^{\perp_{\mathbb{R}}}\,,$$

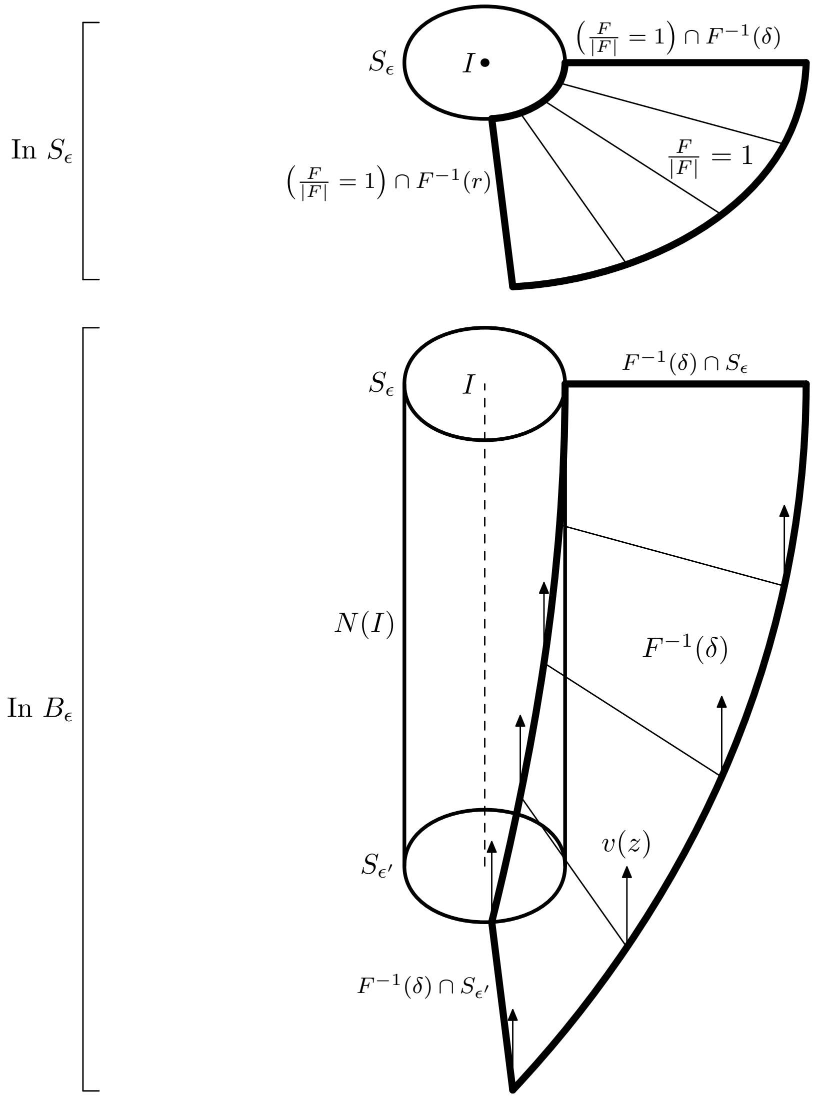

FIGURE 3. Vector field and indetermination points

be the real orthogonal complement of the line spanned by the vector $u(z)$.

Since $\dim_{\mathbb{R}} Q = 2n-1$ and $\dim_{\mathbb{R}} T = 2n-1$, the real vector space $Q \cap T$ has dimension at least $n-2$. Let $\pi : \mathbb{R}^{2n} \to Q \cap T$ be the orthogonal projection on $Q \cap T$ in the direction of the vector $iu(z)$. We set

$$v_3(z) = \pi(u(z))\,.$$

Obviously $v_3(z) \in T$ and an easy computation shows that v_3 verifies conditions (i) and (iii).

Last, let us consider $U_4 = V \setminus M(F)$. Let $z \in U_4$. We set $\gamma(z) = |f(z)|^2 + |g(z)|^2$. There exists a vector $v_4(z)$ verifying (i), (ii) and

$v_4(z) \in T$ if and only if the vector

$$w(z) = \operatorname{grad}_{\mathbb{R}} \gamma(z)$$

does not belong to the complex vector space H generated by the two vectors $w_1(z) = z$ and $w_2(z) = \operatorname{grad} F(z)$. This is equivalent to saying $z \notin N(F)$, which is true because F is semitame and (i)-tame.

Now, we define globally the vector field on X by a partition of unity. □

Third step : integration of the vector field v.

We integrate the vector field v and we denote by $C = \{z = p(t)\}$ an integral curve.

Condition (i) implies that the argument of $F(p(t))$ is constant and that and that $|F(p(t))|$ is strictly increasing along C. Conditions (ii) and (iii) implies that $\|p(t)\|$ is strictly increasing along C.

Lemma 17. *If C pass through a point $z_0 \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{S}_{\varepsilon'})$, then C reaches $\mathbb{S}_\varepsilon$ at a point z_1 such that $|F(z_1)| = r$.*

Proof. Let C' be the arc of curve in $\mathbb{R}^2$ parameterized by $t \in [0, \log(r/\delta)]$ as follows :

- $x(t) = t + \log \delta$
- $\forall t \in [0, \log(\rho/\delta)], y(t) = 2b_1 t + \varepsilon'^2$
- $\forall t \in [\log(\rho/\delta), \log(r/\delta)], y(t) = 2b_2 t + \varepsilon''^2$

The arc C' is the union of the two segments joining the three points $(\log \delta, \varepsilon'^2)$, $(\log \rho, \varepsilon''^2)$ and $(\log r, \varepsilon^2)$. Then C' is included in the zone $P = [\log \delta, \log r] \times [0, {\varepsilon'_0}^2] \cup [\log(r/2), \log r] \times [0, \varepsilon^2]$ and $C' \cap \psi(U) = \emptyset$. (see Figure 4). Now, let C be an integral curve of v passing through $z_0 \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{S}_{\varepsilon'}$. Then a computation analogous to that of [8] page 53, shows that C' is nothing but the image of C by ψ. Therefore, the integral curve C passing through z_0 goes transversally to the spheres centered at 0 until it reaches $\mathbb{S}_\varepsilon$ at a point belonging to $F^{-1}(\mathbb{S}^1_r(0))$. □

Then, the diffeomorphism

$$\Theta_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)),$$

which sends $z \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'})$ on the intersection $\Theta_0(z)$ of the integral curve of v passing through z with the sphere $S_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0))$, is a diffeomorphism from the fibration:

$$F : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0)$$

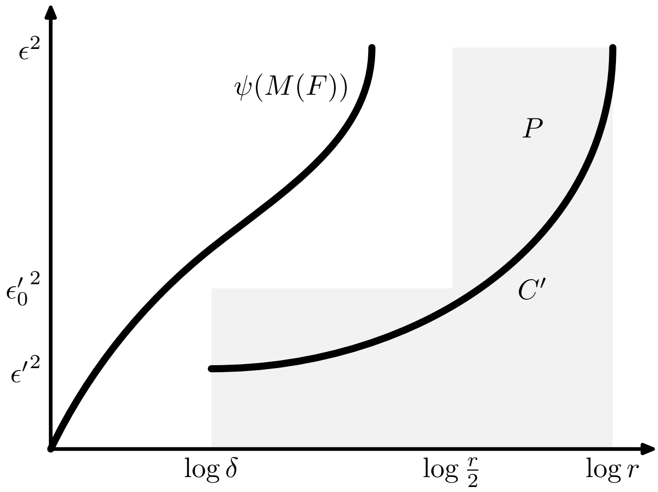

FIGURE 4. Avoidance of $M(F)$

to the fibration:

$$\Phi = \frac{F}{|F|} : \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)) \longrightarrow \mathbb{S}^1. \tag{7}$$

This completes the proof of the theorem.

8. AN EXAMPLE

Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be the two holomorphic germs defined by:

$$f(x, y) = x^2 + y^3, \quad g(x, y) = x^2 + y^3.$$

Let $\pi : X \longrightarrow U$ be the resolution of the meromorphic function $F = f/g$ whose divisor is represented on Figure 1. On Figure 5 we draw its dual graph G. The numbers between parentheses are the multiplicities of F along the corresponding component of the total transform of fg by π, *i.e.*, the $(m_i^f - m_i^g)$ where m_i^f and m_i^g are the multiplicities of $f \circ \pi$ and $g \circ \pi$. The numbers without parentheses are the Euler classes. The arrows are for the strict transforms of f and g and for the strict transform of a generic fibre of F.

The meromorphic function f/g is semitame, and (i)-tame ($n = 2$). One therefore has three different fibrations: the global Milnor fibration of f/g,

$$\Phi_F = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1,$$

and the two local Milnor fibrations $\phi_0 = F_| : F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(0)$ and $\phi_\infty = F_| : F^{-1}(\mathbb{S}^1_\delta(\infty)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(\infty)$

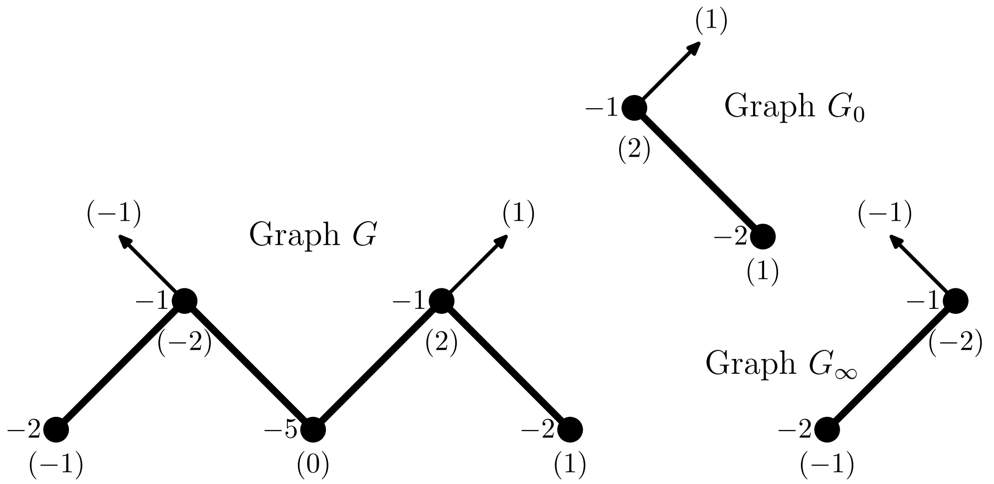

FIGURE 5. Resolution of $x^2 + y^3/x^3 + y^2/g$

Using the fibration theorem for plumbed multilinks (see e.g. [13, 2.11]), one observes three different fibred multilinks in plumbing manifolds on this configuration :

(1) The link $L_f - L_g$ in the sphere $\mathbb{S}^3$.
(2) The link L_f in the plumbed manifold V_0 whose graph G_0 is the subgraph of G determined by the divisor $E_2 \cup E_3$.
(3) The link L_g in the plumbed manifold V_∞ whose graph G_∞ is the subgraph of G determined by the divisor $E_3 \cup E_4$.

As already mentioned, the map Φ_F is a fibration of the link $L_f - L_g$ in the sphere $\mathbb{S}^3$. The two local fibrations ϕ_0 and ϕ_∞ are the restrictions to the complementary of the indetermination set I of f/g of fibrations $\hat{\phi}_0$ and $\hat{\phi}_\infty$ of the links $L_f \subset V_0$ and $L_g \subset V_\infty$.

The fibres $\hat{\mathcal{M}}_F^0$ and $\hat{\mathcal{M}}_F^\infty$ of $\hat{\phi}_0$ and $\hat{\phi}_\infty$ can be computed by the Hurwitz formula from the graphs G_0 and G_∞. One obtains for both a sphere with one hole. The fibre $\mathcal{M}_F^0$ (resp. $\mathcal{M}_F^\infty$) is then obtained by removing a neighbourhood of the intersection of $\hat{\mathcal{M}}_F^0$ with $\pi^{-1}(0)$. One then obtains a sphere with 3 holes in both cases. Now the fiber of Φ_F is homeomorphic to the surface obtained by glueing together $\mathcal{M}_F^0$ and $\mathcal{M}_F^\infty$ along the two boundary components just created.

At last, let us recall that the isomorphism classes of the fibrations $\Phi_F, \hat{\phi}_0$ and $\hat{\phi}_\infty$ are completely described by the Nielsen invariants of their monodromies $h : \mathcal{M}_F \to \mathcal{M}_F$, $\hat{h}_0 : \hat{\mathcal{M}}_F^0 \to \hat{\mathcal{M}}_F^0$ and $\hat{h}_\infty : \hat{\mathcal{M}}_F^\infty \to \hat{\mathcal{M}}_F^\infty$ (see e.g. [11]). The set of Nielsen invariants is equivalent to the data of the graph G, G_0 and G_∞ respectively, weighted by the genus and the multiplicities.

In particular, it should be mentioned that the Dehn twist performed by h on the two glueing curves $\partial_0 \subset \mathcal{M}_F$ is positive (it equals $+\frac{5}{2}$ according, for instance, to the formula (3) of [11, Lemma 4.4]), whereas all the Dehn twists performed by the monodromy of the Milnor fibration associated with a holomorphic germ are negative. This is a general phenomena in the case $n = 2$: each dicritical component of the exceptional divisor in the resolution of f/g gives rise to a positive Dehn twist along each of the corresponding separating curves on the fiber, see [2, Chapter 5].

The previous arguments show that in this example, the monodromy of the global Milnor fibration of the meromorphic germ f/g can not possibly be the monodromy of a holomorphic germ. Then, a natural question is to ask whether something similar happens in higher dimensions. That is, is there some property that distinguishes the monodromy of the global Milnor fibration of a meromorphic function from those of holomorphic germs?

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Integral points on generic fibers

INTEGRAL POINTS ON GENERIC FIBERS

ARNAUD BODIN

Abstract. Let $P(x,y)$ be a rational polynomial and $k \in \mathbb{Q}$ be a generic value. If the curve $(P(x,y)=k)$ is irreducible and admits an infinite number of points whose coordinates are integers then there exist algebraic automorphisms that send $P(x,y)$ to the polynomial x or to x^2-dy^2, $d \in \mathbb{N}$. Moreover for such curves (and others) we give a sharp bound for the number of integral points (x,y) with x and y bounded.

1. Introduction

Let $P \in \mathbb{Q}[x,y]$ be a polynomial and $\mathcal{C} = (P(x,y)=0) \subset \mathbb{C}^2$ be the corresponding algebraic curve. On old and famous result is the following:

Theorem (Siegel's theorem)**.** *Suppose that $\mathcal{C}$ is irreducible. If the number of integral points $\mathcal{C} \cap \mathbb{Z}^2$ is infinite then $\mathcal{C}$ is a rational curve.*

Our first goal is to prove a stronger version of Siegel's theorem for curve defined by an equation $\mathcal{C} = (P(x,y)=k)$ where k is a generic value. More precisely there exists a finite set $\mathcal{B}$ such that the topology of the complex plane curve $(P(x,y)=k) \subset \mathbb{C}^2$ is independent of $k \notin \mathcal{B}$. We say that $k \notin \mathcal{B}$ is a *generic value*.

Theorem 1. *Let $P \in \mathbb{Q}[x,y]$ and let $k \in \mathbb{Q} \setminus \mathcal{B}$ be a generic value. Suppose that the algebraic curve $\mathcal{C} = (P(x,y)=k)$ is irreducible. If $\mathcal{C}$ contains an infinite number of integral points $(m,n) \in \mathbb{Z}^2$ then there exists an algebraic automorphism $\Phi \in \operatorname{Aut}\mathbb{Q}^2$ such*

$$P \circ \Phi(x,y) = x \quad \text{or} \quad P \circ \Phi(x,y) = \alpha(x^2-dy^2)+\beta,$$

where $d \in \mathbb{N}^*$ is a non-square and $\alpha \in \mathbb{Q}^*$, $\beta \in \mathbb{Q}$.

In particular the curve $\mathcal{C} = (P(x,y)=k)$ is diffeomorphic to a line $(x=0)$, in which case the set $\mathcal{B}$ is empty or to an hyperbola $x^2-dy^2=1$ in which case $\mathcal{B}$ is a singleton.

Theorem 1 can be seen as an arithmetic version of the Abhyankar-Moh-Suzuki theorem [2] and in fact we use this result. It can also be seen as a strong version of a result of Nguyen Van Chau [8] concerning counter-examples to the Jacobian conjecture.

For example, let $P(x,y) = x - y^d$. The curve $\mathcal{C} = (x - y^d = 0)$ has infinitely many integral points of type (n^d, n), $n \in \mathbb{Z}$. And for the algebraic automorphism $\Phi(x,y) = (x + y^d, y)$ we have $P \circ \Phi(x,y) = x$. In particular the curve $\mathcal{C}$ is send (by Φ^{-1}) to $(x = 0)$. As pointing out by Kevin Buzzard the second case corresponds to Pell's equation and for example $(x^2 - 2y^2 = 1)$ admits an infinite number of integral points. Of course a kind of reciprocal of Theorem 1 is true. Let $Q_1(x,y) = x$, (resp. $Q_2(x,y) = x^2 - dy^2$) and $\Phi \in \operatorname{Aut} \mathbb{Q}^2$ whose inverse Φ^{-1} as integral coefficients. If we set $P_1 = Q_1 \circ \Phi$ (resp. $P_2 = Q_2 \circ \Phi$) then the curve $(P_1 = 0)$ (resp. $(P_2 = 1)$) has infinitely many integral points.

For non-generic values the result is not true, for example let $P(x,y) = x^2 - y^3$ and $\mathcal{C} = (x^2 - y^3 = 0)$. The integral points (n^3, n^2), $n \in \mathbb{Z}$ belongs to $\mathcal{C}$, but as $\mathcal{C}$ is singular it cannot be algebraically equivalent to a line.

We will apply Theorem 1 to obtain new bounds for the number of integral points on algebraic curves. Let $\mathcal{C} = (P(x,y) = 0)$ be an algebraic curve, and let $d = \deg P$. Let

$$N(\mathcal{C}, B) = \# \left\{ (x,y) \in \mathcal{C} \cap \mathbb{Z}^2 \mid |x| \leqslant B \text{ and } |y| \leqslant B \right\}.$$

For all irreducible curves Heath-Brown [5] proved that $N(\mathcal{C}, B) \leqslant C_d B^{\frac{1}{d}+\varepsilon}$ for some constant C_d. By making the proof of [5] explicit Walkowiak obtains in [12]:

Theorem (Explicit Heath-Brown's theorem). *For all irreducible curve $\mathcal{C}$ of degree d and all $B > 0$:*

$$N(\mathcal{C}, B) \leqslant 2^{48} d^8 \ln(B)^5 B^{\frac{1}{d}}.$$

The term $B^{\frac{1}{d}}$ in Theorem 2 is sharp but the term $2^{48} d^8 \ln(B)^5$ is far from being optimal. For curves $\mathcal{C}$ as in Theorem 1 we will give sharp bounds for $N(\mathcal{C}, B)$. First of all if $\mathcal{C} = (P = k)$ and the polynomial P is algebraically equivalent to $x^2 - dy^2$ it is know [11, p. 135] that there exists $C > 0$ and $n \geqslant 1$ such that

$$N(\mathcal{C}, B) \leqslant C \cdot \ln(B)^n.$$

Of course it implies $N(\mathcal{C}, B) \leqslant B^{\frac{1}{d}}$ for sufficiently large B and we shall omit this case.

So if P is not algebraically equivalent to $x^2 - dy^2$ then as a corollary of Theorem 1, such a curve $\mathcal{C}$ admits a parametrisation by polynomials: let $(p(t), q(t))$ be a parametrisation of $\mathcal{C}$ with rational coefficients.

Moreover $\deg P$ is equal to $\deg p$ or $\deg q$. We suppose $\deg P = \deg p$ and write

$$p(t) = \frac{1}{b}(a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0).$$

where $a_0, \ldots, a_d, b \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_d, b) = 1$.

Theorem 2. *Let $\mathcal{C}$ be as in Theorem 1. Then the number $N(\mathcal{C}, B)$ of integral points on $\mathcal{C}$ bounded by B, verifies:*

$$\forall \varepsilon > 0 \quad \exists B_0 \geqslant 0 \quad \forall B \geqslant B_0 \qquad N(\mathcal{C}, B) \leqslant 2a_d^{1-\frac{1}{d}} b^{\frac{1}{d}} B^{\frac{1}{d}} + 1 + \varepsilon.$$

In fact by adding a term $\frac{(d-1)(d-2)}{2}$ we prove such a bound for curves $\mathcal{C}$ parametrised by polynomials $(p(t), q(t))$.

For example if $a_d = 1$ and $b = 1$ in Theorem 2 we get for a sufficiently large B:

$$N(\mathcal{C}, B) \leqslant 2B^{\frac{1}{d}} + 1 + \varepsilon.$$

It implies for $\varepsilon = \frac{1}{2}$ and all sufficiently large B such that $B^{\frac{1}{d}}$ is an integer that we get

$$N(\mathcal{C}, B) \leqslant 2B^{\frac{1}{d}} + 1.$$

For instance we have a parametrisation of $\mathcal{C} = (x - y^d = 0)$ by (t^d, t). If $B^{\frac{1}{d}}$ is an integer we get $N(\mathcal{C}, B) = 2B^{\frac{1}{d}} + 1$. Moreover the "$\varepsilon$" term is necessary as the example $\mathcal{C} = (x - y^d = 1)$ proves, see Example 13.

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2. Parametrisation

2.1. **Topology of polynomials.** By a result of Thom for a polynomial $P \in \mathbb{Q}[x, y]$ seen as a map $P : \mathbb{C}^2 \to \mathbb{C}$ there exists a finite set $\mathcal{B} \subset \mathbb{C}$ such that

$$P : P^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$$

is a topological locally trivial fibration. A value $k \notin \mathcal{B}$ is a *generic value.*

For example we have the following characterization of the generic values: the Euler characteristic of the complex plane curve $(P(x, y) = k) \subset \mathbb{C}^2$ is independent of $k \notin \mathcal{B}$ and jumps if and only if $k \in \mathcal{B}$.

Of course the image by P of a singular point is not a generic value, but for example $P(x, y) = x(xy - 1)$ has no singular points while $\mathcal{B} = \{0\}$. Then if $k \notin \mathcal{B}$ is a generic value the plane algebraic curve $\mathcal{C} =$

$(P(x,y) = k) \subset \mathbb{C}^2$ does not have singular points. Moreover if $\mathcal{C}$ is connected then $\mathcal{C}$ is irreducible.

The connectedness of a generic fiber $\mathcal{C}$ is equivalent of $P(x,y)$ being non-composite [1]. We recall that $P(x,y)$ is *composite* if there exist $h \in \mathbb{C}[t]$, $\deg h \geqslant 2$, and $Q \in \mathbb{C}[x,y]$ such that $P(x,y) = h \circ Q(x,y)$. By [3, Theorem 7] we even can choose h and Q with rational coefficients. Consequently it has been noticed by Janusz Gwozdziewicz that the hypothesis "$\mathcal{C}$ is irreducible" in Theorem 1 can be removed. In that case the conclusion becomes $P \circ \Phi(x,y) = h(x)$ or $P \circ \Phi(x,y) = h(x^2 - dy^2)$, where $h \in \mathbb{Q}[t]$ is a one-variable polynomial of positive degree.

2.2. **Algebraic automorphisms.** For $K = \mathbb{Q}$ or $K = \mathbb{C}$ an *algebraic automorphism* $\Phi \in \operatorname{Aut} K^2$ is a polynomial map $\Phi : K^2 \longrightarrow K^2$, invertible, such that the inverse is also a polynomial map. The polynomials $P, Q \in K[x,y]$ are *algebraically equivalent* if there exists $\Phi \in \operatorname{Aut} K^2$ such that $Q = P \circ \Phi$. And in fact such P and Q have the same topological and algebraic properties.

2.3. **Siegel's theorem.** By Siegel's theorem an irreducible plane algebraic curve $\mathcal{C}$ with an infinite number of integral points $(x,y) \in \mathcal{C} \cap \mathbb{Z}^2$ is a rational curve (i.e. the genus is zero).

2.4. **Parametrisation.** As $\mathcal{C} = (P(x,y) = k)$ is rational it admits a parametrisation by rational fractions. In order to deals with points at infinity and special parameters we compactify the situation.

Let $\bar{P}(x,y,z)$ be the homogenisation of $P(x,y)$ with $d = \deg \bar{P} = \deg P$. Then $\bar{\mathcal{C}} = (P(x,y,z) - kz^d = 0) \subset \mathbb{P}^2$ is the closure of $\mathcal{C}$.

As $\bar{\mathcal{C}}$ is rational there exists a birational map $\phi : \mathbb{P}^1 \longrightarrow \bar{\mathcal{C}}$ defined by

$$\phi(t,s) = (\bar{p}(t,s), \bar{q}(t,s), \bar{r}(t,s))$$

where $\bar{p}, \bar{q}, \bar{r}$ are homogeneous polynomials of the same degree in $\mathbb{Z}[t,s]$, without common non-constant factor.

We will need some facts about parametrisations (see [9]).

Lemma 3. *For any such parametrisation:*

(1) *ϕ is a morphism (it is well-defined everywhere);*
(2) *ϕ is surjective;*
(3) $\deg \bar{p} = \deg \bar{q} = \deg \bar{r} = \deg P = d$*;*
(4) *The birational inverse ψ of ϕ is well-defined away from the singular points of $\bar{\mathcal{C}}$;*
(5) *If $(x,y) \in \mathcal{C} \cap \mathbb{Z}^2$ is a non-singular point then there exists $(t,s) \in \mathbb{Z}^2$ such that $\phi(t,s) = (x : y : 1)$;*

Proof. We just sketch the proofs and we refer to [9, Lemma 2.1] for details and references: the fact that $\bar{p}, \bar{q}, \bar{r}$ have no common factor implies that ϕ is well-defined. Then $\phi(\mathbb{P}^1)$ is dense in $\bar{\mathcal{C}}$ and hence is equal to $\bar{\mathcal{C}}$. The birational inverse ψ of ϕ is well-defined excepted at the singular points of $\bar{\mathcal{C}}$, see Fulton [4, p.160]. Then a non-singular integral point of $\bar{\mathcal{C}}$ is send by ψ to a point with rational coordinates. We will not need item (3) and we refer to [9]. □

2.5. **Maillet's theorem.** An old result of Maillet [6] gives strong restrictions for the parametrisations.

Theorem 4. *If $\mathcal{C} = (P(x,y) = k) \subset \mathbb{C}^2$ has an infinite number of integral points then there exists a parametrisation ϕ of $\bar{\mathcal{C}}$ given by*

$$\phi(t,s) = (\bar{p}(t,s), \bar{q}(t,s), \bar{r}(t,s))$$

as before with

$$\bar{r}(t,s) = at^d \quad \text{or} \quad \bar{r}(s,t) = a(\alpha t^2 + \beta ts + \gamma s^2)^{d/2}$$

where $a, \alpha, \beta, \gamma \in \mathbb{Z}$ and $\beta^2 - 4\alpha\gamma > 0$.

2.6. **Topology of $\mathcal{C}$.**

Lemma 5. *If k is a generic value and $\mathcal{C} = (P(x,y) = k)$ has an infinite number of integral points then $\mathcal{C}$ is homeomorphic to $\mathbb{C}$ or to $\mathbb{C}^*$.*

In fact the homeomorphisms are diffeomorphisms and we only need k to be a non-singular value of $P(x,y)$. In Lemma 9 we will exclude the case $\mathbb{C}^*$.

Proof. Let $L_\infty = (z = 0)$ of $\mathbb{P}^2$ be the line at infinity. What are the parameters $(t:s)$ that give the points at infinity $\bar{\mathcal{C}} \cap L_\infty$? The points at infinity correspond to the parameters $(t:s)$ such that $\bar{r}(s,t) = 0$.

So if ϕ is the parametrisation given by Maillet's theorem then for $\bar{r}(t,s) = at^d$ then $(t:s) = (0:1)$ and there is one point $P \in \bar{\mathcal{C}} \cap L_\infty$. For $\bar{r}(s,t) = a(\alpha t^2 + \beta ts + \gamma s^2)^{d/2}$ then $(t:s) = (\tau_1 : 1)$ or $(t,s) = (\tau_2 : 1)$ where τ_1, τ_2 are the roots of $\alpha t^2 + \beta t + \gamma$ (say $\alpha \neq 0$); there are two points $P_1, P_2 \in \bar{\mathcal{C}} \cap L_\infty$.

From Lemma 3 we know that ϕ is morphism that induces a bijective map onto $\bar{\mathcal{C}} \setminus \operatorname{Sing} \bar{\mathcal{C}}$. The only singular points of $\bar{\mathcal{C}}$ are on the line at infinity L_∞ because $\mathcal{C}$ is a non-singular affine curve. Then in the case $\bar{r}(s,t) = at^d$, the restriction $\phi_| : \mathbb{P}^1 \setminus \{(0:1)\} \longrightarrow \bar{\mathcal{C}} \setminus \{P\}$ is a bijective map. Moreover it is an homeomorphism. But $\mathbb{P}^1 \setminus \{(0:1)\} = \mathbb{C}$ and $\bar{\mathcal{C}} \setminus \{P\} = \mathcal{C}$. Then $\mathcal{C}$ is homeomorphic to $\mathbb{C}$.

In the case $\bar{r}(s,t) = a(\alpha t^2 + \beta ts + \gamma s^2)^{d/2}$, the restriction $\phi_| : \mathbb{P}^1 \setminus \{(\tau_1 : 1), (\tau_2 : 1)\} \longrightarrow \bar{\mathcal{C}} \setminus \{P_1, P_2\}$ is an homeomorphism from $\mathbb{P}^1 \setminus \{(\tau_1 : 1), (\tau_2 : 1)\}$ to $\mathcal{C} = \bar{\mathcal{C}} \setminus \{P_1, P_2\}$. Then $\mathcal{C}$ is homeomorphic to $\mathbb{C}^*$. □

2.7. **Case of $\mathcal{C}$ being homeomorphic to $\mathbb{C}$.** We recall the Abhyankar-Moh-Suzuki [1, 2, 10] theorem:

Theorem 6.

(1) *Let $t \mapsto (p(t), q(t))$ be an injective polynomial map over $\mathbb{C}$ such that the tangent vector $(p'(t), q'(t))$ is never $(0,0)$ then $\deg p$ divides $\deg q$, or $\deg q$ divides $\deg p$.*

(2) *Let $\mathcal{C} = (P(x,y) = 0) \subset \mathbb{C}^2$ be an algebraic plane curve non-singular and homeomorphic to $\mathbb{C}$ then there exists an algebraic automorphism $\Phi \in \operatorname{Aut} \mathbb{C}^2$ such that*

$$P \circ \Phi(x,y) = x.$$

The second statement is the usual form of the Abhyankar-Moh-Suzuki theorem, it is in fact a consequence of the first (see the proof below), for which a more general statement exists [2].

Lemma 7. *If $\mathcal{C} = (P(x,y) = k)$, k a generic value, is homeomorphic to $\mathbb{C}$ then there exists $\Phi \in \operatorname{Aut} \mathbb{Q}^2$ whose inverse has integral coefficients such that $P \circ \Phi(x,y) = ax + b$, $a, b \in \mathbb{Q}$.*

Proof. The existence of $\Phi \in \operatorname{Aut} \mathbb{C}^2$ is Theorem 6-(2). But here we ask the coefficients of Φ to be rationals and those of Φ^{-1} to be integers: we will apply Theorem 6-(1). As $\mathcal{C}$ is homeomorphic to $\mathbb{C}$ let $(p(t), q(t))$ be a parametrisation of $\mathcal{C}$ with $p(t), q(t) \in \mathbb{Q}[t]$ (it comes from setting $s = 1$ in a parametrisation $\phi = (\bar{p}(t,s), \bar{q}(t,s), as^d)$ of $\bar{\mathcal{C}}$). Then $\deg p$ divides $\deg q$ or $\deg q$ divides $\deg p$. Suppose that $\delta = \deg p > 0$ divides $\deg q$ and write $p(t) = a_\delta t^\delta + a_{\delta-1} t^{\delta-1} + \cdots$ and $q(t) = b_{\ell\delta} t^{\ell\delta} + b_{\ell\delta-1} t^{\ell\delta-1} + \cdots$ with $a_i, b_i \in \mathbb{Q}$ and $\ell \geqslant 1$. Write $a_\delta = \frac{\alpha}{\beta}$ and $b_{\ell\delta} = \frac{\alpha'}{\beta'}$. Set the algebraic automorphism of $\operatorname{Aut} \mathbb{Q}^2$,

$$\Phi_1(x,y) = \left(\frac{1}{\alpha\beta} x, \frac{1}{\alpha} \ell y - \frac{\beta^\ell}{\alpha^\ell \beta'} x^\ell \right),$$

whose inverse is

$$\Phi_1^{-1}(x,y) = \left(\alpha\beta' x, \alpha' y + \alpha' {\beta'}^{\ell-1} \beta^\ell x^\ell \right),$$

whose coefficients are integers.

The composition with Φ_1 yields a parametrisation of $(P-k) \circ \Phi_1(x,y)$ given by $p(t), q'(t)$ with $q'(t) \in \mathbb{Q}[t]$ and $\deg q' < \deg q$. We repeat this process until one of $p(t)$ or $q(t)$ is a constant the other being of degree 1 (because $\mathcal{C}$ does not have singular points). Then by the algebraic automorphism $\Phi = \Phi_1 \circ \Phi_2 \circ \cdots$, whose inverse has integral coefficients, we get $(P - k) \circ \Phi(x,y) = ax + b$, $a, b \in \mathbb{Q}$. $\square$

2.8. **Case of $\mathcal{C}$ being homeomorphic to $\mathbb{C}^*$.** We will need the classification over $\mathbb{C}$ of polynomials with a generic fiber homeomorphic to $\mathbb{C}^*$, due to W. Neumann [7, §8].

Theorem 8. *If $\mathcal{C} = (P(x,y) = k)$, $k \neq 0$ a generic value, is homeomorphic to $\mathbb{C}^*$ then P there exists an algebraic automorphism $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such that*

$$P \circ \Phi(x,y) = x^p y^q + \beta$$

$$\text{or} \qquad P \circ \Phi(x,y) = x^p(yx^r + a_{r-1}x^{r-1} + \cdots + a_0)^q + \beta,$$

with $\beta \in \mathbb{C}$, $p > 0$, $q > 0$, $\gcd(p,q) = 1$, $r > 0$, $a_0, \ldots, a_{r-1} \in \mathbb{C}$ and $a_0 \neq 0$.

We will prove that only some special polynomials of the first type can have an infinite number of integral points.

The main result of this part is the following lemma.

Lemma 9. *If $\mathcal{C} = (P(x,y) = k)$, k a generic value, is homeomorphic to $\mathbb{C}^*$ with an infinite number of integral points then there exists $\Phi \in \operatorname{Aut}\mathbb{Q}^2$ such that $P \circ \Phi(x,y) = \alpha(x^2 - dy^2) + \beta$, $\alpha \in \mathbb{Q}^*$, $\beta \in \mathbb{Q}$.*

Proof. By Theorem 8 we know that the polynomial P has exactly two absolute irreducible factors. For simplicity of the redaction we suppose in the following that $\beta = 0$.

First case : P is reducible in $\mathbb{Q}[x,y]$.

Once again we will prove that Φ of Theorem 8 can be chosen with rational coefficients and its inverse with integral coefficients. We write $P = \alpha A^p B^q$ the decomposition into irreducible factors with $A, B \in \mathbb{Q}[x,y]$. Again for simplicity we suppose $\alpha = 1$. We will decompose the proof according to the cases of Theorem 8. In both cases we see that the curve $(P = 0)$ has a non-singular irreducible component homeomorphic to $\mathbb{C}$ (the one send by Φ^{-1} to $(x = 0)$). Notice that this curve $(P = 0)$ is *not* the curve $\mathcal{C}$. This component homeomorphic to $\mathbb{C}$ is either $(A = 0)$ or $(B = 0)$, say it is $(A = 0)$. Then, as $A \in \mathbb{Q}[x,y]$, by the version of Abhyankar-Moh-Suzuki theorem used as in Lemma 7 above, there exists $\Psi \in \operatorname{Aut}\mathbb{Q}^2$, whose inverse has integral coefficients, such that: $A \circ \Psi(x,y) = ax + b$, this implies :

$$P \circ \Psi(x,y) = (ax + b)^p Q(x,y)^q.$$

Sub-case $P \circ \Phi(x,y) = x^p y^q$.

Then $(Q(x,y) = 0)$ is non-singular, homeomorphic to $\mathbb{C}$ and the intersection multiplicity with $ax + b$ is 1. Then if $(p(t), q(t))$ is a polynomial parametrisation of $(Q(x,y) = 0)$ we have $\deg p = 1$. Then as in the proof of Lemma 7 by algebraic automorphisms of type $(x,y) \mapsto$

$(\alpha x, \beta y - \gamma x^\ell)$ whose inverse have integral coefficients, we can suppose that $q(t)$ is a constant. Notice that such automorphisms preserve vertical lines.

Then $Q(x,y)$ becomes $cy+d$, $c,d \in \mathbb{Q}$, while $ax+b$ remains unchanged. Then we have find $\Psi' \in \operatorname{Aut}\mathbb{Q}^2$ whose inverse has integral coefficients such that:

$$P \circ \Psi'(x,y) = (ax+b)^p(cy+d)^q.$$

Sub-case $P \circ \Phi(x,y) = x^p(yx^r + a_{r-1}x^{r-1} + \cdots + a_0)^q$**.**

$P \circ \Phi(x,y) = x^p(yx^r + a_{r-1}x^{r-1} + \cdots + a_0)^q$ is algebraically equivalent to $P \circ \Psi(x,y) = (ax+b)^p Q(x,y)^q$ by the algebraic automorphism $\Phi \circ \Psi^{-1}$. Moreover $\Phi \circ \Psi^{-1}$ should send x to $ax+b$. Then $\Phi \circ \Psi^{-1}$ is the composition of algebraic automorphisms of type $(x,y) \mapsto (ax+b,y)$ and $(x,y) \mapsto (\alpha x, \beta y - \gamma x^\ell)$. This implies that the degree in the variable y remains unchanged. Then $\deg_y Q(x,y) = \deg_y(yx^r + a_{r-1}x^{r-1} + \cdots + a_0) = 1$. Then $Q(x,y) = q_1(x)y + q_2(x)$. Due to the asymptotic branches we have $q_1(x,y) = (ax+b)^r$. And by algebraic automorphisms whose inverse have integral coefficients of type $(x,y) \mapsto (\alpha x, \beta y - \gamma x^\ell)$ we can suppose $\deg q_2 < r$. Then we have found $\Psi' \in \operatorname{Aut}\mathbb{Q}^2$ with an inverse having integral coefficients such that:

$$P \circ \Psi'(x,y) = (ax+b)^p(y(ax+b)^r + b_{r-1}x^{r-1} + \cdots + b_0)^q,$$

$b_0, \ldots, b_{r-1} \in \mathbb{Q}$, $b_0 \neq 0$.

Conclusion for both sub-cases.

Now the curve $(P \circ \Psi'(x,y) = k)$ has a finite number of integral points since the branches at infinity are asymptotic to horizontal or vertical lines (with equation $(ax+b=0)$, $(cy+d=0)$ in the first case and $(ax+b=0)$, $(y=0)$ in the second case). Now as Ψ'^{-1} has integral coefficients, an integral point $(m,n) \in (P(x,y)=k) \cap \mathbb{Z}^2$ is sent to an integral point $\Psi'^{-1}(m,n) \in (P \circ \Psi'(x,y) = k) \cap \mathbb{Z}^2$ it implies that $\mathcal{C} = (P(x,y)=k)$ also have a finite number of integral points.

Second case : P **is irreducible in** $\mathbb{Q}[x,y]$**.**

Then by Lemma 10 below it implies that there exist $D, E \in \mathbb{Q}[x,y]$, $d \in \mathbb{Z}$ such that $P = C^2 - dD^2$. Then $P = (C - \sqrt{d}D)(C + \sqrt{d}D)$ is the decomposition into irreducible factors.

Sub-case $P \circ \Phi(x,y) = x^p y^q$**.**

The by Lemma 10 we know that $p=1$, $q=1$. And equivalently there exists $\Phi' \in \operatorname{Aut}\mathbb{C}^2$ such that $P \circ \Phi'(x,y) = (x - \sqrt{d}y)(x + \sqrt{d}y)$. Then $P \circ \Phi'(x,y) = (C^2 - dD^2) \circ \Phi'(x,y) = (x - \sqrt{d}y)(x + \sqrt{d}y)$. We may suppose that $(C - \sqrt{d}D) \circ \Phi'(x,y) = (x - \sqrt{d}y)$ and $(C + \sqrt{d}D) \circ \Phi'(x,y) = (x + \sqrt{d}y)$, by addition and subtraction we get

$C \circ \Phi'(x,y) = x$ and $D \circ \Phi'(x,y) = y$. Then $(CD) \circ \Phi'(x,y) = xy$, with $C, D \in \mathbb{Q}[x,y]$. As in the first case above we are now enable to find $\Psi \in \operatorname{Aut}\mathbb{Q}^2$ (whose inverse as integral coefficients) such that $C \circ \Psi(x,y) = x$, $D \circ \Psi(x,y) = y$ and $CD \circ \Psi(x,y) = xy$. Now $P \circ \Psi(x,y) = (C^2 - dD^2) \circ \Psi(x,y) = x^2 - dy^2$.

Sub-case $P \circ \Phi(x,y) = x^p(yx^r + a_{r-1}x^{r-1} + \cdots + a_0)^q$.

Again $p = 1$, $q = 1$, and we may suppose that $(C - \sqrt{d}D) \circ \Phi(x,y) = x$. We denote $Q = y \circ \Phi^{-1}(x,y)$ i.e. $Q \circ \Phi(x,y) = y$. Then

$$\begin{aligned}(C - \sqrt{d}D)(C + \sqrt{d}D) = P& \\ &= x(yx^r + a_{r-1}x^{r-1} + \cdots + a_0) \circ \Phi^{-1}(x,y) \\ &= (C - \sqrt{d}D)\big(Q(C - \sqrt{d}D)^r + \cdots\big)\end{aligned}$$

Then

$$(C + \sqrt{d}D) = \big(Q(C - \sqrt{d}D)^r + \cdots\big).$$

But as $C, D \in \mathbb{Q}[x,y]$ we have $d = \deg(C + \sqrt{d}D) = \deg(C - \sqrt{d}D)$ and we get $d = \deg Q + rd$. As $r \geqslant 1$ we get $\deg Q = 0$ which is in contradiction with the definition of Q. Then this sub-case does not occur. ☐

Lemma 10. *Let $P \in \mathbb{Q}[x,y]$ such that $P = \alpha A^p B^q$, with $\gcd(p,q) = 1$ and with $\alpha \in \mathbb{Q}^*$, $A, B \in \overline{\mathbb{Q}}[x,y]$ normalized and irreducible (that is to say P admits exactly two absolute irreducible factors). Then either $A, B \in \mathbb{Q}[x,y]$ or $p = 1$, $q = 1$ and there exist $C, D \in \mathbb{Q}[x,y]$, $d \in \mathbb{Z}$ non-square such that $P = \alpha(C^2 - dD^2)$.*

The following proof is due to Pierre Dèbes.

Proof. Let $a_{i,j} \in \overline{\mathbb{Q}}$ be the coefficients of A. Let n be the degree of the finite extension $\mathbb{Q}((a_{i,j}))/\mathbb{Q}$. Then there exist exactly n distinct conjugates of A. But for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma(A) \in \{A, B\}$. Then A has at most two distinct algebraic conjugates. Thus $n = 1$ or $n = 2$. If $A \notin \mathbb{Q}[x,y]$ then there exists $a_{i_0,j_0} \notin \mathbb{Q}$ and a $\sigma_0 \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma_0(A) = B$. Then $n = 2$ so that the extension $\mathbb{Q}((a_{i,j}))/\mathbb{Q}$ is quadratic. Moreover $p = q = 1$. This implies the existence of a non-square integer d such that $A, B \in \mathbb{Q}(\sqrt{d})[x,y]$. Now if we write $A = C + \sqrt{d}D$, $C, D \in \mathbb{Q}[x,y]$ then its algebraic conjugate is $B = C - \sqrt{d}D$. ☐

Lemma 7 and Lemma 9 imply Theorem 1 of the introduction.

3. Number of integral points on polynomials curves

Lemma 11. *Let* $p(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0 \in \mathbb{Q}[t]$, $a_d > 0$. *Let* $\sigma = -\frac{a_{d-1}}{da_d}$. *For all* $\varepsilon > 0$ *there exists* $B_0 > 0$ *such that for all* $B \geqslant B_0$ *and*

$$t_+ = \left(\frac{B}{a_d}\right)^{\frac{1}{d}} + \sigma + \varepsilon, \quad t_- = -\left(\frac{B}{a_d}\right)^{\frac{1}{d}} + \sigma - \varepsilon,$$

then

$$\forall t \geqslant t_+ \quad |p(t)| > B \qquad \text{and} \qquad \forall t \leqslant t_- \quad |p(t)| > B.$$

A similar result holds if $a_d < 0$.

Proof. We write $t = s + \sigma + \varepsilon$ and we look at the asymptotic behaviour for $p(t)$ when t (and s) is large.

$$\begin{aligned} p(t) &= p(s + \sigma + \varepsilon) \\ &= a_d(s + \sigma + \varepsilon)^d + a_{d-1}(s + \sigma + \varepsilon)^{d-1} + \cdots \\ &= a_d s^d + (da_d(\sigma + \varepsilon) + a_{d-1})s^{d-1} + o(s^{d-1}) \\ &= a_d s^d + da_d\varepsilon s^{d-1} + o(s^{d-1}). \end{aligned}$$

For $s = \left(\frac{B}{a_d}\right)^{\frac{1}{d}}$ then $a_d s^d = B$ we have

$$\begin{aligned} p(t_+) &= p(s + \sigma + \varepsilon) \\ &= B \cdot \left(1 + \varepsilon d\frac{1}{s} + o\left(\frac{1}{s}\right)\right). \end{aligned}$$

Then for all sufficiently large B (such that $s > 0$ is large enough) we have $p(t_+) \geqslant B\left(1 + \frac{\varepsilon d}{2}\frac{1}{s}\right)$ then $p(t_+) > B$. Now the function $t \mapsto p(t)$ is an increasing function for sufficiently large t. Then for all sufficiently large B: if $t \geqslant t_+$ then $p(t) \geqslant p(t_+) > B$.

Now

$$\begin{aligned} p(t_-) &= p(-s + \sigma - \varepsilon) \\ &= (-1)^d B \cdot \left(1 + \varepsilon d\frac{1}{s} + o\left(\frac{1}{s}\right)\right). \end{aligned}$$

Then for all sufficiently large B, $|p(t_-)| > B$. And again if $t < t_-$ then $|p(t)| \geqslant |p(t_-)| > B$. ☐

For a polynomial $p(t) \in \mathbb{Q}[t]$ in one variable we defined:

$$M(p, B) = \{t \in \mathbb{Q} \mid p(t) \in \mathbb{Z} \text{ and } |p(t)| \leqslant B\}.$$

Lemma 12. *Let* $p(t) = \frac{1}{b}(a_d t^d + \cdots + a_0) \in \mathbb{Q}[t]$*,* $a_0, \ldots, a_d, b \in \mathbb{Z}$*,* $\gcd(a_0, \ldots, a_d, b) = 1$ *and* $a_d > 0$*. Then for all* $\varepsilon > 0$ *there exists* $B_0 > 0$ *such that for all* $B \geqslant B_0$ *we have*

$$M(p, B) \leqslant 2a_d^{1-\frac{1}{d}} b^{\frac{1}{d}} B^{\frac{1}{d}} + 1 + \varepsilon.$$

Proof. If $t = \frac{\alpha}{\beta} \in \mathbb{Q}$ and $p(\frac{\alpha}{\beta}) = k \in \mathbb{Z}$ then it is well-known that β divides a_d. Then such t belongs to $\frac{1}{a_d}\mathbb{Z}$. Let $\varepsilon > 0$ and let B_0 be as in Lemma 11. Again by Lemma 11 if $t > 0$ and $|p(t)| \leqslant B$ then $t < t_+ = \left(\frac{B}{a_d/b}\right)^{\frac{1}{d}} + \sigma + \varepsilon$. If $t < 0$ and $|p(t)| \leqslant B$ then $|t| < |t_-| = -t_- = \left(\frac{B}{a_d/b}\right)^{\frac{1}{d}} - \sigma + \varepsilon$. Now the cardinal of $\frac{1}{a_d}\mathbb{Z} \cap [t_-, t_+]$ is less than $a_d|t_+| + a_d|t_-| + 1 = 2a_d \left(\frac{B}{a_d/b}\right)^{\frac{1}{d}} + \sigma - \sigma + 2\varepsilon + 1 = 2a_d \left(\frac{bB}{a_d}\right)^{\frac{1}{d}} + 1 + 2\varepsilon$. $\square$

Of course if $p(t)$ is a monic polynomial with integral coefficients, i.e. $b = 1$, $a_d = 1$, then $M(p, B) \leqslant 2B^{\frac{1}{d}} + 1 + \varepsilon$. For example if $p(t) = t^d$ then $M(p, B) = 2B^{\frac{1}{d}} + 1$. The following examples show that the "ε" is necessary and that the bound of Lemma 12 is the best one (at least for $a_d = 1$).

Example 13. Let $p(t) = t^d - 1$ where d is an even number. Then the following assertion:

$$\exists B_0 > 0 \quad \forall B \geqslant B_0 \quad M(p, B) \leqslant 2B^{\frac{1}{d}} + 1 \quad \text{is false.}$$

In fact for k any positive integer, set $B_k = p(k) = k^d - 1$. Then as d is even for all $t \in [-k, k]$ we have $t^d - 1 \leqslant k^d - 1 = B_k$ then $M(p, B_k) = 2k + 1$. Now if the assertion were true we would have $M(p, B_k) \leqslant 2B_k^{\frac{1}{d}} + 1$ that it to say $2k + 1 \leqslant 2p(k)^{\frac{1}{d}} + 1$, then $k^d \leqslant p(k) = k^d - 1$ which gives the contradiction.

Here is another example.

Example 14. Let $p(t) = \frac{1}{d!}(t-1)(t-2)\cdots(t-d)$ then for all $t \in \mathbb{Z}$ we have $p(t) \in \mathbb{Z}$. Conversely if $p(\frac{\alpha}{\beta}) \in \mathbb{Z}$ then $\beta = 1$. Then for a positive integer k and $B_k = p(k)$, $|p(-k+d)| \leqslant B_k$. we have $M(p, B_k) \geqslant 2k + 1 - d$. Lemma 12 applied with $a_d = 1$ and $b = d!$ gives $M(p, B) \leqslant 2(d!)^{\frac{1}{d}} B_k^{\frac{1}{d}} + 1 + \varepsilon$. Now $B_k = p(k) \leqslant \frac{k^d}{d!}$ and we get

$$2k + 1 - d \leqslant M(p, B_k) \leqslant 2k + 1 + \varepsilon.$$

We apply these computations to the situation of our curves.

Let $P(x, y) \in \mathbb{Q}[x, y]$ be irreducible, let $\mathcal{C} = (P(x, y) = 0)$. Then $\mathcal{C}$ is a *polynomial curve* if it admits a polynomial parametrisation

$(p(t), q(t))$, $p(t), q(t) \in \mathbb{Q}[t]$. Equivalently $\mathcal{C}$ is a rational curve with one place at infinity. Moreover $\deg P = \max(\deg p, \deg q)$. We will suppose $\deg P = \deg p$ and we write $p(t) = \frac{1}{b}(a_d t^d + \cdots + a_0)$ as before.

Lemma 15. *Let $\mathcal{C}$ be a polynomial curve. Suppose* $\deg P = d = \deg p$, $p(t) = \frac{1}{b}(a_d t^d + \cdots + a_0)$. *Then for all $\varepsilon > 0$ there exists $B_0 > 0$ such that for all $B \geqslant B_0$:*

$$N(\mathcal{C}, B) \leqslant 2a_d^{1-\frac{1}{d}} b^{\frac{1}{d}} B^{\frac{1}{d}} + \frac{(d-1)(d-2)}{2} + 1 + \varepsilon.$$

The term $\frac{(d-1)(d-2)}{2}$ comes from the number of singular points; for non-singular curves we get the bound of Theorem 2.

Proof. An algebraic curve of degree d must have less than $\frac{(d-1)(d-2)}{2}$ singular points, see [4, p.117]. The other integral points $(p(t), q(t))$ of $\mathcal{C}$ correspond to rational parameters t, see Lemma 3. Now we apply Lemma 12. □

Examples 13 and 14 give polynomials parametrised by $(p(t), t)$ that proves that the bound of Lemma 15 is asymptotically sharp.

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Reducibility of rational functions in several variables

REDUCIBILITY OF RATIONAL FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. We prove a analogous of Stein theorem for rational functions in several variables: we bound the number of reducible fibers by a formula depending on the degree of the fraction.

1. INTRODUCTION

Let K be an algebraically closed field. Let $f = \frac{p}{q} \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$, $n \geqslant 2$ and $\gcd(p,q) = 1$, the *degree* of f is $\deg f = \max\{\deg p, \deg q\}$. We associate to a fraction $f = \frac{p}{q}$ the pencil $p - \lambda q$, $\lambda \in \hat{K}$ (where we denote $\hat{K} = K \cup \{\infty\}$ and by convention if $\lambda = \infty$ then $p - \lambda q = q$).

For each $\lambda \in \hat{K}$ write the decomposition into irreducible factors:

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_i^{r_i}.$$

The *spectrum* of f is $\sigma(f) = \{\lambda \in \hat{K} \mid n_\lambda > 1\}$, and the *order of reducibility* is $\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1)$.

A fraction f is *composite* if it is the composition of a univariate rational fraction of degree more than 1 with another rational function.

Theorem 1.1. *Let K be an algebraically closed field of characteristic 0. Let $f \in K(\underline{x})$ be non-composite then*

$$\rho(f) < (\deg f)^2 + \deg f.$$

A theorem of Bertini and Krull implies that if f is non-composite then $\sigma(f)$ is finite and we should notice that $\#\sigma(f) \leqslant \rho(f)$. Later on, for an algebraically closed field of characteristic zero and for a polynomial $f \in K[x,y]$, Stein [St] proved the formula $\rho(f) < \deg f$. This formula has been generalized in several directions, see [Na1] for references. For a rational function $f \in \mathbb{C}(x,y)$ a consequence of the work of Ruppert [Ru] on pencil of curves, is that $\#\sigma(f) < (\deg f)^2$. For K algebraically closed (of any characteristic) and $f \in K(x,y)$ Lorenzini [Lo] proved under geometric hypotheses on the pencil $(p - \lambda q)$ that

$\rho(f) < (\deg f)^2$. This has been generalized by Vistoli [Vi] for a pencil in several variables for an algebraically closed field of characteristic 0.

Let us give an example extracted from [Lo]. Let $f(x,y) = \frac{x^3+y^3+(1+x+y)^3}{xy(1+x+y)}$, then $\deg(f) = 3$ and $\sigma(f) = \{1, j, j^2, \infty\}$ (where $\{1, j, j^2\}$ are the third roots of unity). For $\lambda \in \sigma(f)$, $(f = \lambda)$ is composed of three lines hence $\rho(f) = 8 = (\deg f)^2 - 1$. Then Lorenzini's bound is optimal in two variables.

The motivation of this work is that we develop the analogous theory of Stein for rational function: composite fractions, kernels of Jacobian derivatives, groups of divisors,... The method for the two variables case is inspired from the work of Stein [St] and the presentation of that work by Najib [Na1]. For completeness even the proofs similar to the ones of Stein have been included. Another motivation is that with a bit more effort we get the case of several variables by following the ideas of [Na1] (see the articles [Na2], [Na3]).

In §2 we prove that a fraction is non-composite if and only its spectrum is finite. Then in §3 we introduce a theory of Jacobian derivation and compute the kernel. Next in §4 we prove that for a non-composite fraction in two variables $\rho(f) < (\deg f)^2 + \deg f$. Finally in §5 we extend this formula to several variables and we end by stating a result for fields of any characteristic.

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2. Composite rational functions

Let K be an algebraically closed field. Let $\underline{x} = (x_1, \ldots, x_n)$, $n \geqslant 2$.

Definition 2.1. A rational function $f \in K(\underline{x})$ is *composite* if there exist $g \in K(\underline{x})$ and $r \in K(t)$ with $\deg r \geqslant 2$ such that

$$f = r \circ g.$$

Theorem 2.2. *Let* $f = \frac{p}{q} \in K(\underline{x})$. *The following assertions are equivalent:*

1. f *is composite;*
2. $p - \lambda q$ *is reducible in* $K[\underline{x}]$ *for all* $\lambda \in \hat{K}$ *such that* $\deg p - \lambda q = \deg f$*;*
3. $p - \lambda q$ *is reducible in* $K[\underline{x}]$ *for infinitely many* $\lambda \in \hat{K}$.

Before proving this result we give two corollaries.

Corollary 2.3. f *is non-composite if and only if its spectrum* $\sigma(f)$ *is finite.*

One aim of this paper is to give a bound for $\sigma(f)$. The hard implication of this theorem (3) $\Rightarrow$ (1) is in fact a reformulation of a theorem of Bertini and Krull.

We also give a nice application pointed out to us by P. Débes:

Corollary 2.4. *Let $p \in K[\underline{x}]$ irreducible. Let $q \in K[\underline{x}]$ with $\deg q < \deg p$ and $\gcd(p,q) = 1$. Then for all but finitely many $\lambda \in K$, $p - \lambda q$ is irreducible in $K[\underline{x}]$.*

Convention : When we define a fraction $F = \frac{P}{Q}$ we will assume that $\gcd(P,Q) = 1$.

We start with the easy part of Theorem 2.2:

Proof. (2) $\Rightarrow$ (3) is trivial. Let us prove (1) $\Rightarrow$ (2). Let $f = \frac{p}{q}$ be a composite rational function. There exist $g = \frac{u}{v} \in K(\underline{x})$ and $r \in K(t)$ with $k = \deg r \geqslant 2$ such that $f = r \circ g$. Let us write $r = \frac{a}{b}$. Let $\lambda \in \hat{K}$ such that $\deg a - \lambda b = \deg r$ and factorize $a(t) - \lambda b(t) = \alpha(t-t_1)(t-t_2)\cdots(t-t_k)$, $\alpha \in K^*, t_1, \ldots, t_k \in K$. Then

$$p - \lambda q = q \cdot (f - \lambda) = q \cdot \left(\frac{a - \lambda b}{b}\right)(g) = \alpha q \frac{(g-t_1)\cdots(g-t_k)}{b(g)}.$$

Then by multiplication by v^k at the numerator and denominator we get:

$$(p - \lambda q) \cdot (v^k b(g)) = \alpha q (u - t_1 v) \cdots (u - t_k v),$$

which is a polynomial identity. As $\gcd(a,b) = 1$, $\gcd(u,v) = 1$ and $\gcd(p,q) = 1$ then $u - t_1 v, \ldots, u - t_k v$ divide $p - \lambda q$. Hence $p - \lambda q$ is reducible in $K[\underline{x}]$. □

Let us reformulate the Bertini-Krull theorem in our context from [Sc, Theorem 37]. It will enable us to end the proof of Theorem 2.2.

Theorem 2.5 (Bertini, Krull)**.** *Let $F(\underline{x},\lambda) = p(\underline{x}) - \lambda q(\underline{x}) \in K[\underline{x},\lambda]$ an irreducible polynomial. Then the following conditions are equivalent:*

(1) *$F(\underline{x},\lambda_0) \in K[\underline{x}]$ is reducible for all $\lambda_0 \in K$ such that $\deg_{\underline{x}} F(\underline{x},\lambda_0) = \deg_{\underline{x}} F$.*

(2) (a) *either there exist $\phi, \psi \in K[\underline{x}]$ with $\deg_{\underline{x}} F > \max\{\deg\phi, \deg\psi\}$, and $a_i \in K[\lambda]$, such that*

$$F(\underline{x},\lambda) = \sum_{i=0}^{n} a_i(\lambda)\phi(\underline{x})^{n-i}\psi(\underline{x})^i;$$

(b) *or $\mathrm{char}(K) = \pi > 0$ and $F(\underline{x},\lambda) \in K[\underline{x}^\pi,\lambda]$, where $\underline{x}^\pi = (x_1^\pi, \ldots, x_n^\pi)$.*

We now end the proof of Theorem 2.2:

Proof. (3) $\Rightarrow$ (1) Suppose that $p - \lambda_0 q$ is reducible in $K[\underline{x}]$ for infinitely many $\lambda_0 \in \hat{K}$; then it is reducible for all $\lambda_0 \in K$ such that $\deg_{\underline{x}} F(\underline{x}, \lambda_0) = \deg_{\underline{x}} F$ (see Corollary 3 of Theorem 32 of [Sc]). We apply Bertini-Krull theorem:

Case (a): $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x})$ can be written:

$$p(\underline{x}) - \lambda q(\underline{x}) = \sum_{i=0}^{n} a_i(\lambda) \phi(\underline{x})^{n-i} \psi(\underline{x})^i.$$

So we may suppose that for $i = 1, \dots, n$, $\deg_\lambda a_i = 1$, let us write $a_i(\lambda) = \alpha_i - \lambda \beta_i$, $\alpha_i, \beta_i \in K$. Then

$$p(\underline{x}) = \sum_{i=0}^{n} \alpha_i \phi(\underline{x})^{n-i} \psi(\underline{x})^i = \phi^n \sum_{i=0}^{n} \alpha_i \left(\frac{\psi}{\phi}\right)^i (\underline{x}),$$

and

$$q(\underline{x}) = \sum_{i=0}^{n} \beta_i \phi(\underline{x})^{n-i} \psi(\underline{x})^i = \phi^n \sum_{i=0}^{n} \beta_i \left(\frac{\psi}{\phi}\right)^i (\underline{x}).$$

If we set $g(\underline{x}) = \frac{\psi(\underline{x})}{\phi(\underline{x})} \in K[\underline{x}]$, and $r(t) = \frac{\sum_{i=0}^{n} \alpha_i t^i}{\sum_{i=0}^{n} \beta_i t^i}$ then $\frac{p}{q}(\underline{x}) = r \circ g$. Moreover as $\deg_{\underline{x}} F > \max\{\deg \phi, \deg \psi\}$ this implies $n \geqslant 2$ so that $\deg r \geqslant 2$. Then $\frac{p}{q} = f = r \circ g$ is a composite rational function

Case (b): Let $\pi = \operatorname{char}(K) > 0$ and $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x}) \in K[\underline{x}^\pi, \lambda]$, For $\lambda = 0$ it implies that $p(\underline{x}) = P(\underline{x}^\pi)$, then there exists $p' \in K[\underline{x}]$ such that $p(\underline{x}) = (p'(\underline{x}))^\pi$. For $\lambda = -1$ we obtain $s' \in K[\underline{x}]$ such that $p(\underline{x}) + q(\underline{x}) = (s'(\underline{x}))^\pi$. Then $q(\underline{x}) = (p(\underline{x}) + q(\underline{x})) - p(\underline{x}) = (s'(\underline{x}))^\pi - (p'(\underline{x}))^\pi = (s'(\underline{x}) - p'(\underline{x}))^\pi$. Then if we set $q' = s' - p'$ we obtain $q(\underline{x}) = (q'(\underline{x}))^\pi$. Now set $r(t) = t^\pi$ and $g = \frac{p'}{q'}$ we get $f = \frac{p}{q} = \left(\frac{p'}{q'}\right)^\pi = r \circ g$. $\square$

3. Kernel of the Jacobian derivation

We now consider the two variables case and K is an uncountable algebraically closed field of characteristic zero.

3.1. **Jacobian derivation.** Let $f, g \in K(x, y)$, the following formula:

$$D_f(g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

defines a derivation $D_f : K(x, y) \to K(x, y)$. Notice the $D_f(g)$ is the determinant of the Jacobian matrix of (f, g). We denote by C_f the kernel of D_f:

$$C_f = \{g \in K(x, y) \mid D_f(g) = 0\}.$$

Then C_f is a subfield of $K(x,y)$. We have the inclusion $K(f) \subset C_f$. Moreover if $g^k \in C_f$, $k \in \mathbb{Z} \setminus \{0\}$ then $g \in C_f$.

Lemma 3.1. *Let $f = \frac{p}{q}$, $g \in K(x,y)$. The following conditions are equivalent:*

1. *$g \in C_f$;*
2. *f and g are algebraically dependent;*
3. *g is constant on irreducible components of the curves $(p - \lambda q = 0)$ for all but finitely many $\lambda \in \hat{K}$;*
4. *g is constant on infinitely many irreducible components of the curves $(p - \lambda q = 0)$, $\lambda \in \hat{K}$.*

Corollary 3.2. *If $g \in C_f$ is not a constant then $C_f = C_g$.*

Proof.

- (1) $\Leftrightarrow$ (2). We follow the idea of [Na1] instead of [St]. f and g are algebraically dependent if and only $\operatorname{transc}_K K(f,g) = 1$. And $\operatorname{transc}_K K(f,g) = 1$ if and only the rank of the Jacobian matrix of (f,g) is less or equal to 1, which is equivalent to $g \in C_f$.
- (2) $\Rightarrow$ (3). Let f and g be algebraically dependent. Then there exists a two variables polynomial in f and g that vanishes. Let us write
$$\sum_{i=0}^{n} R_i(f) g^i = 0$$
where $R_i(t) \in K[t]$. Let us write $f = \frac{p}{q}$, $g = \frac{u}{v}$ and $R_n(t) = \alpha(t-\lambda_1)\cdots(t-\lambda_m)$. Then
$$\sum_{i=0}^{n} R_i\left(\frac{p}{q}\right)\left(\frac{u}{v}\right)^i = 0, \text{ hence } \sum_{i=0}^{n} R_i\left(\frac{p}{q}\right) u^i v^{n-i} = 0.$$
By multiplication by q^d for $d = \max\{\deg R_i\}$ (in order that $q^d R_i(\frac{p}{q})$ are polynomials) we obtain
$$q^d R_n\left(\frac{p}{q}\right) u^n = v\left(-q^d R_{n-1}\left(\frac{p}{q}\right) u^{n-1} - \cdots\right).$$
As $\gcd(u,v) = 1$ then v divides the polynomial $q^d R_n(\frac{p}{q})$, then v divides $q^{d-m}(p - \lambda_1 q)\cdots(p - \lambda_m q)$. Then all irreducible factors of v divide q or $p - \lambda_i q$, $i = 1, \ldots, m$.
  Let $\lambda \notin \{\infty, \lambda_1, \ldots, \lambda_m\}$. Let V_λ be an irreducible component of $p - \lambda q$, then $V_\lambda \cap Z(v)$ is zero dimensional (or empty). Hence

v is not identically equal to 0 on V_λ. Then for all but finitely many $(x,y) \in V_\lambda$ we get:

$$\sum_{i=0}^{n} R_i(\lambda) g(x,y)^i = 0.$$

Therefore g can only reach a finite number of values $c_1, \ldots, c_n$ (the roots of $\sum_{i=0}^{n} R_i(\lambda)t^i$). Since V_λ is irreducible, g is constant on V_λ.

- (3) $\Rightarrow$ (4). Clear.
- (4) $\Rightarrow$ (1). We first give a proof that if g is constant along an irreducible component V_λ of $(p - \lambda q = 0)$ then $D_f(g) = 0$ on V_λ (we suppose that V_λ is not in the poles of g). Let $(x_0, y_0) \in V_\lambda$ and $t \mapsto p(t)$ be a local parametrization of V_λ around (x_0, y_0). By definition of $p(t)$ we have $f(p(t)) = \lambda$, this implies that:

$$\left\langle \frac{dp}{dt} \mid \overline{\operatorname{grad} f} \right\rangle = \frac{d(f(p(t))}{dt} = 0$$

and by hypotheses g is constant on V_λ this implies $g(p(t))$ is constant and again:

$$\left\langle \frac{dp}{dt} \mid \overline{\operatorname{grad} g} \right\rangle = \frac{d(g(p(t))}{dt} = 0.$$

Then $\operatorname{grad} f$ and $\operatorname{grad} g$ are orthogonal around (x_0, y_0) on V_λ to the same vector, as we are in dimension 2 this implies that the determinant of Jacobian matrix of (f, g) is zero around (x_0, y_0) on V_λ. By extension $D_f(g) = 0$ on V_λ.

We now end the proof: If g is constant on infinitely many irreducible components V_λ of $(p - \lambda q = 0)$ this implies that $D_f(g) = 0$ on infinitely many V_λ. Then $D_f(g) = 0$ in $K(x,y)$.

□

3.2. **Group of the divisors.** Let $f = \frac{p}{q}$, let $\lambda_1, \ldots, \lambda_n \in \hat{K}$, we denote by $G(f; \lambda_1, \ldots, \lambda_n)$ the multiplicative group generated by all the divisors of the polynomials $p - \lambda_i q$, $i = 1, \ldots, n$.

Let

$$d(f) = (\deg f)^2 + \deg f.$$

Lemma 3.3. *Let $F_1, \ldots, F_r \in G(f; \lambda_1, \ldots, \lambda_n)$. If $r \geqslant d(f)$ then there exists a collection of integers $m_1, \ldots, m_r$ (not all equal to zero) such that*

$$g = \prod_{i=1}^{r} F_i^{m_i} \in C_f.$$

Proof. Let $\mu \notin \{\lambda_1, \dots, \lambda_n\}$, and let S be an irreducible component of $(p - \mu q = 0)$. Let $\bar{S}$ be the projective closure of S. The functions F_i restricted to $\bar{S}$ have their poles and zeroes on the points at infinity of S or on the intersection $S \cap Z(F_i) \subset Z(p) \cap Z(q)$.

Let $n : \tilde{S} \to \bar{S}$ be a normalization of $\bar{S}$. The inverse image under normalisation of the points at infinity are denoted by $\{\gamma_1, \dots, \gamma_k\}$, their number verifies $k \leqslant \deg S \leqslant \deg f$.

At a point $\delta \in Z(p) \cap Z(q)$, the number of points of $n^{-1}(\delta)$ is the local number of branches of S at δ then it is less or equal than $\mathrm{ord}_\delta(S)$, where $\mathrm{ord}_\delta(S)$ denotes the order (or multiplicity) of S at δ (see e.g. [Sh], paragraph II.5.3). Then

$$\begin{aligned}\#n^{-1}(\delta) &\leqslant \mathrm{ord}_\delta(S) \leqslant \mathrm{ord}_\delta Z(p - \mu q) \leqslant \mathrm{ord}_\delta Z(p - \mu q) \cdot \mathrm{ord}_\delta Z(p) \\ &\leqslant \mathrm{mult}_\delta(p - \mu q, p) = \mathrm{mult}_\delta(p, q)\end{aligned}$$

where $\mathrm{mult}_\delta(p, q)$ is the intersection multiplicity (see e.g. [Fu]). Then by Bézout theorem:

$$\sum_{\delta \in Z(p) \cap Z(q)} \#n^{-1}(\delta) \leqslant \sum_{\delta \in Z(p) \cap Z(q)} \mathrm{mult}_\delta(p, q) \leqslant \deg p \cdot \deg q \leqslant (\deg f)^2.$$

Then the inverse image under normalisation of $\cup_{i=1}^{r} S \cap Z(F_i)$ denoted by $\{\gamma_{k+1}, \dots, \gamma_\ell\}$ have less or equal than $(\deg f)^2$ elements. Notice that $\ell \leqslant \deg f + (\deg f)^2 = d(f)$.

Now let ν_{ij} be the order of F_i at γ_j ($i = 1, \dots, r$; $j = 1, \dots, \ell$). Consider the matrix $M = (\nu_{ij})$. Because the degree of the divisor (F_i) (seen over $\tilde{S}$) is zero we get $\sum_{j=1}^{\ell} \nu_{ij} = 0$, for $i = 1, \dots, r$, that means that columns of M are linearly dependent. Then $\mathrm{rk}\, M < \ell \leqslant d(f)$, by hypothesis $r \geqslant d(f)$, then the rows of M are also linearly dependent. Let $m_1(\mu, S), \dots, m_r(\mu, S)$ such that $\sum_{i=1}^{r} m_i(\mu, S)\nu_{ij} = 0$, $j = 1, \dots, \ell$.

Consider the function $g_{\mu,S} = \prod_{i=1}^{r} F_i^{m_i(\lambda,S)}$. Then this function is regular and does not have zeroes or poles at the points γ_j, because $\sum_{i=1}^{r} m_i(\mu, S)\nu_{ij} = 0$. Then $g_{\mu,S}$ is constant on S.

This construction gives a map $(\mu, S) \mapsto (m_1(\mu, S), \dots, m_r(\mu, S))$ from K to $\mathbb{Z}^r$. Since K is uncountable, there exists infinitely many (μ, S) with the same $(m_1, \dots, m_r)$. Then the function $g = \prod_{i=1}^{r} F_i^{m_i}$ is constant on infinitely many components of curves of $(p - \mu q = 0)$ and by Lemma 3.1 this implies $g \in C_f$. □

3.3. **Non-composite rational function.** Let $f = \frac{p}{q}$. Let $G(f)$ be the multiplicative group generated by all divisors of the polynomials

$p-\lambda q$ for all $\lambda \in \hat{K}$. In fact we have

$$G(f) = \bigcup_{(\lambda_1,\ldots,\lambda_n)\in K^n} G(f;\lambda_1,\ldots,\lambda_n).$$

Definition 3.4. A family $F_1,\ldots,F_r \in G(f)$ is *f-free* if $(m_1,\ldots,m_r) \in \mathbb{Z}^r$ is such that $\prod_{i=1}^r F_i^{m_i} \in C_f$ then $(m_1,\ldots,m_r) = (0,\ldots,0)$.

A f-free family $F_1,\ldots,F_r \in G(f)$ is *f-maximal* if for all $F \in G(f)$, $\{F_1,\ldots,F_r,F\}$ is not f-free.

Theorem 3.5. *Let $f \in K(x,y)$, $\deg f > 0$. Then the following conditions are equivalent:*

(1) $\deg f = \min\{\deg g \mid g \in C_f \setminus K\}$*;*
(2) $\sigma(f)$ *is finite;*
(3) $C_f = K(f)$*;*
(4) f *is non-composite.*

Remark 3.6. This does not give a new proof of "$\sigma(f)$ is finite $\Leftrightarrow$ f is non-composite" because we use Bertini-Krull theorem.

Remark 3.7. The proof (1) $\Rightarrow$ (2) is somewhat easier than in [St], whereas (2) $\Rightarrow$ (3) is more difficult.

Proof.

- (1) $\Rightarrow$ (2). Let us suppose that $\sigma(f)$ is infinite. Set $f = \frac{p}{q}$, with $\gcd(p,q) = 1$. For all $\alpha \in \sigma(f)$, let F_α be an irreducible divisor of $p - \alpha q$, such that $\deg F_\alpha < \deg f$. By Lemma 3.3 there exists a f-maximal family $\{F_1,\ldots,F_r\}$ with $r \leqslant d(f)$. Moreover $r \geqslant 1$ because $\{F_\alpha\}$ is f-free: if not there exists $k \neq 0$ such that $F_\alpha^k \in C_f$ then $F_\alpha \in C_f$, but $\deg F_\alpha < \deg f$ that contradicts the hypothesis of minimality.

  Now the collection $\{F_1,\ldots,F_r,F_\alpha\}$ is not f-free, so that there exist integers $\{m_1(\alpha),\ldots,m_r(\alpha),m(\alpha)\}$, with $m(\alpha) \neq 0$, such that
  $$F_1^{m_1(\alpha)} \cdots F_r^{m_r(\alpha)} \cdot F_\alpha^{m(\alpha)} \in C_f.$$
  Since $\sigma(f)$ is infinite then is equal to $\hat{K}$ minus a finite number of values (see Theorem 2.2) then $\sigma(f)$ is uncountable and the map $\alpha \mapsto (m_1(\alpha),\ldots,m_r(\alpha),m(\alpha))$ is not injective. Let $\alpha \neq \beta$ such that $m_i(\alpha) = m_i(\beta) = m_i$, $i = 1,\ldots,r$ and $m(\alpha) = m(\beta) = m$. Then $F_1^{m_1}\cdots F_r^{m_r} \cdot F_\alpha^m \in C_f$ and $F_1^{m_1}\cdots F_r^{m_r} \cdot F_\beta^m \in C_f$, it implies that $(F_\alpha/F_\beta)^m \in C_f$, therefore $F_\alpha/F_\beta \in C_f$.

  Now $\deg \frac{F_\alpha}{F_\beta} < \deg f$, then by the hypothesis of minimality it proves $\frac{F_\alpha}{F_\beta}$ is a constant. Let $a \in K^*$ such that $F_\alpha = aF_\beta$, by definition F_α divides $p-\alpha q$, but moreover F_α divides $p-\beta q$ (as F_β do). Then as F_α divides both $p-\alpha q$ and $p-\beta q$, F_α divides p and q, that contradicts $\gcd(p,q)=1$.

- $(2) \Rightarrow (3)$. Let $f=\frac{p}{q}$, $\sigma(f)$ finite and $g \in C_f$, we aim at proving that $g \in K(f)$. The proof will be done in several steps:
  (a) *Reduction to the case $g=\frac{u}{q^\ell}$.* Let $g=\frac{u}{v} \in C_f$, then f and g are algebraically dependent, then there exists a polynomial in f and g that vanishes. As before let us write

$$\sum_{i=0}^{n} R_i(f) g^i = 0$$

where $R_i(t) \in K[t]$. As $f=\frac{p}{q}$, $g=\frac{u}{v}$ then

$$\sum_{i=0}^{n} R_i\Big(\frac{p}{q}\Big)\left(\frac{u}{v}\right)^i = 0, \text{ hence } \sum_{i=0}^{n} R_i\Big(\frac{p}{q}\Big) u^i v^{n-i} = 0.$$

By multiplication by q^d for $d=\max\{\deg R_i\}$ (in order that all $q^d R_i(\frac{p}{q})$ are polynomials) we get:

$$q^d R_n\Big(\frac{p}{q}\Big)u^n = v\left(-q^d R_{n-1}\Big(\frac{p}{q}\Big)u^{n-1} - \cdots\right).$$

As $\gcd(u,v)=1$ then v divides the polynomial $q^d R_n(\frac{p}{q})$; we write $vu' = q^d R_n(\frac{p}{q})$ then

$$g = \frac{u}{v} = \frac{uu'}{q^d R_n(\frac{p}{q})}.$$

But $R_n(\frac{p}{q}) \in K(\frac{p}{q})$ then $\frac{uu'}{q^d} \in C_f$, but also we have that $g \in K(f)$ if and only if $\frac{uu'}{q^d} \in K(f)$. This proves the reduction.

  (b) *Reduction to the case $g=qu$.* Let $g=\frac{u}{q^\ell} \in C_f$, $\ell \geqslant 0$. As $\sigma(f)$ is finite by Lemma 3.1 we choose $\lambda \in K$ such that $p-\lambda q$ is irreducible and $g \in C_f$ is constant (equal to c) on $p-\lambda q$. As $g=\frac{u}{q^\ell}$,we have $p-\lambda q$ divides $u-cq^\ell$. We can write:

$$u - cq^\ell = u'(p-\lambda q).$$

Then

$$\frac{u}{q^\ell} = \frac{u'}{q^{\ell-1}}\Big(\frac{p}{q}-\lambda\Big)+c.$$

As $\frac{u}{q^\ell}$ and $f = \frac{p}{q}$ are in C_f we get $\frac{u'}{q^{\ell-1}} \in C_f$; moreover $\frac{u}{q^\ell} \in K(f)$ if and only if $\frac{u'}{q^{\ell-1}} \in K(f)$. By induction on $\ell \geqslant 0$ this prove the reduction.

(c) *Reduction to the case $g = q$.* Let $g = qu \in C_f$. g is constant along the irreducible curve $(p - \lambda q = 0)$. Then $qu = u_1(p - \lambda q) + c_1$.
Let $\deg p = \deg q$. Then $q^h u^h = u_1^h(p^h - \lambda q^h)$ (where P^h denotes the homogeneous part of higher degree of the polynomial P). Then $p^h - \lambda q^h$ divides $q^h u^h$ for infinitely many $\lambda \in K$. As $\gcd(p,q) = 1$ this gives a contradiction.
Hence $\deg p \neq \deg q$. We may assume $\deg p > \deg q$ (otherwise $qu \in C_f$ and $\frac{p}{q} \in C_f$ implies $pu \in C_f$). Then we write:

$$qu = qu_1\Big(\frac{p}{q} - \lambda\Big) + c_1,$$

that proves that $qu_1 \in C_f$ and that $qu \in K(f)$ if and only if $qu_1 \in K(f)$. The inequality $\deg p > \deg q$ implies that $\deg u_1 < \deg u$. We continue by induction, $qu_1 = qu_2(\frac{p}{q} - \lambda) + c_2$, with $\deg u_2 < \deg u_1$,..., until we get $\deg u_n = 0$ that is $u_n \in K^*$. Thus we have prove firstly that $qu_n \in C_f$, that is to say $q \in C_f$, and secondly that $qu \in K(f)$ if and only if $q \in K(f)$.

(d) *Case $g = q$.* If $q \in C_f$ then q is constant along the irreducible curve $(p - \lambda q = 0)$ then $q = a(p - \lambda q) + c$, $a \in K^*$. Then

$$q = \frac{c}{1 - a(\frac{p}{q} - \lambda)} \in K\Big(\frac{p}{q}\Big) = K(f).$$

- (3) $\Rightarrow$ (4). Let us assume that $C_f = K(f)$ and that f is composite, then there exist $r \in K(t)$, $\deg r \geqslant 2$ and $g \in K(x,y)$ such that $f = r \circ g$. By the formula $\deg f = \deg r \cdot \deg g$ we get $\deg f > \deg g$. Now if $r = \frac{a}{b}$ then we have a relation $b(g)f = a(g)$, then f and g are algebraically dependent, hence by Lemma 3.1, $g \in C_f$. As $C_f = K(f)$, there exists $s \in K(t)$ such that $g = s \circ f$. Then $\deg g \geqslant \deg f$. That yields to a contradiction.
- (4) $\Rightarrow$ (1). Assume that f is non-composite and let $g \in C_f$ of minimal degree. By Corollary 3.2 we get $C_f = C_g$, then $\deg g = \min\{\deg h \mid h \in C_g \setminus K\}$. Then by the already proved implication (1) $\Rightarrow$ (3) for g, we get $C_g = K(g)$. Then $f \in C_f = C_g = K(g)$, then there exists $r \in K(t)$ such that $f = r \circ g$, but

as f is non-composite then $\deg r = 1$, hence $\deg f = \deg g = \min\{\deg h \mid h \in C_f \setminus K\}$.

□

4. Order of reducibility of rational functions in two variables

Let $f = \frac{p}{q} \in K(x,y)$; for all $\lambda \in \hat{K}$, let n_λ be the number of irreducible components of $p - \lambda q$. Let

$$\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1).$$

By Theorem 2.2, $\rho(f)$ is finite if and only if f is non-composite. We give a bound for $\rho(f)$. Recall that we defined:

$$d(f) = (\deg f)^2 + \deg f.$$

Theorem 4.1. *Let K be an algebraic closed field of characteristic 0. If $f \in K(x,y)$ is non-composite then*

$$\rho(f) < d(f).$$

Proof. First notice that K can be supposed uncountable, otherwise it can be embedded into an uncountable field L and the spectrum in K would be included in the spectrum in L.

Let us assume that f is non-composite, then by Theorem 2.2 and its corollary we have that $\sigma(f)$ is finite: $\sigma(f) = \{\lambda_1, \ldots, \lambda_r\}$. We suppose that $\rho(f) \geqslant d(f)$. Let $f = \frac{p}{q}$. We decompose the polynomials $p - \lambda_i q$ in irreducible factors, for $i = 1, \ldots, r$:

$$p - \lambda_i q = \prod_{j=1}^{n_i} F_{i,j}^{k_{i,j}},$$

where n_i stands for n_{λ_i}. Notice that since $\gcd(p,q) = 1$ then $F_{i,j}$ divides $p - \lambda_i q$ but do not divides any of $p - \mu q$, $\mu \neq \lambda_i$. The collection $\{F_{1,1}, \ldots, F_{1,n_1-1}, \ldots, F_{r,1}, \ldots, F_{r,n_r-1}\}$, is included in $G(f, \lambda_1, \ldots, \lambda_r)$ and contains $\rho(f) \geqslant d(f)$ elements, then Lemma 3.3 provides a collections $\{m_{1,1}, \ldots, m_{1,n_1-1}, \ldots, m_{r,1}, \ldots, m_{r,n_r-1}\}$ of integers (not all equal to 0) such that

$$g = \prod_{i=1}^{r} \prod_{j=1}^{n_i-1} F_{i,j}^{m_{i,j}} \in C_f. \tag{1}$$

By Theorem 3.5 it implies that $g \in K(f)$, then $g = \frac{u(f)}{v(f)}$, where $u, v \in K[t]$. Let $\mu_1, \ldots, \mu_k$ be the roots of u and $\mu_{k+1}, \ldots, \mu_\ell$ the roots of v. Then

$$g = \frac{u(\frac{p}{q})}{v(\frac{p}{q})} = \alpha \frac{\prod_{i=1}^{k} \frac{p}{q} - \mu_i}{\prod_{i=k+1}^{\ell} \frac{p}{q} - \mu_i}$$

so that

$$g = \alpha q^{\ell-2k} \frac{\prod_{i=1}^{k} p - \mu_i q}{\prod_{i=k+1}^{\ell} p - \mu_i q}. \tag{2}$$

If $m_{i_0,j_0} \neq 0$ then by the definition of g by equation (1) and by equation (2), we get that F_{i_0,j_0} divides one of the $p - \mu_i q$ or divides q. If F_{i_0,j_0} divides $p - \mu_i q$ then $\mu_i = \lambda_{i_0} \in \sigma(f)$. If F_{i_0,j_0} divides q then $\mu_i = \infty$, so that $\infty \in \sigma(f)$. In both cases $p - \lambda_{i_0} q$ appears in formula (2) at the numerator or at the denominator of g. Then $F_{i_0,n_{i_0}}$ should appears in decomposition (1), that gives a contradiction. Then $\rho(f) < d(f)$. □

5. Extension to several variables

We follows the lines of the proof of [Na3]. We will need a result that claims that the irreducibility and the degree of a family of polynomials remain constant after a generic linear change of coordinates. For $\underline{x} = (x_1, \dots, x_n)$ and a matrix $B = (b_{ij}) \in Gl_n(K)$, we denote the new coordinates by $B \cdot \underline{x}$:

$$B \cdot \underline{x} = (\sum_{j=1}^{n} b_{1j} x_j, \dots, \sum_{j=1}^{n} b_{nj} x_j).$$

Proposition 5.1. *Let K be an infinite field. Let $n \geqslant 3$ and $p_1, \dots, p_\ell \in K[x_1, \dots, x_n]$ be irreducible polynomials. Then there exists a matrix $B \in Gl_n(K)$ such that for all $i = 1, \dots, \ell$ we get:*

- *$p_i(B \cdot \underline{x})$ is irreducible in $\overline{K(x_1)}[x_2, \dots, x_n]$;*
- $\deg_{(x_2,\dots,x_n)} p_i(B \cdot \underline{x}) = \deg_{(x_1,\dots,x_n)} p_i$.

The proof of this proposition can be derived from [Sm, Ch. 5, Th. 3D] or by using [FJ, Prop. 9.31]. See [Na3] for details.

Now we return to our main result.

Theorem 5.2. *Let K be an algebraically closed field of characteristic 0. Let $f \in K(\underline{x})$ be non-composite then $\rho(f) < (\deg f)^2 + \deg f$.*

Proof. We will prove this theorem by induction on the number n of variables. For $n = 2$, we proved in Theorem 4.1 that $\rho(f) < (\deg f)^2 + \deg f$.

Let $f = \frac{p}{q} \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$. We suppose that f is non-composite. For each $\lambda \in \sigma(f)$ we decompose $p - \lambda q$ into irreducible factors:

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_{\lambda,i}^{r_{\lambda,i}}. \tag{3}$$

We fix $\mu \notin \sigma(f)$. We apply Proposition 5.1 to the polynomials $p-\mu q$ and $F_{\lambda,i}$, for all $\lambda \in \sigma(f)$ and all $i = 1, \ldots, n_\lambda$. Then the polynomials $p(B\cdot\underline{x})-\mu q(B\cdot\underline{x})$ and $F_{\lambda,i}(B\cdot\underline{x})$ are irreducible in $\overline{K(x_1)}[x_2, \ldots, x_n]$ and their degrees in $(x_2, \ldots, x_n)$ are equals to the degrees in $(x_1, \ldots, x_n)$ of $p - \mu q$ and $F_{\lambda,i}$.

Let denote by $k = \overline{K(x_1)}$. This is an uncountable field, algebraically closed of characteristic zero. Now $p(B \cdot \underline{x}) - \mu q(B \cdot \underline{x})$ is irreducible, then $f(B \cdot \underline{x})$ is non-composite in $k(x_2, \ldots, x_n)$.

Now equation (3) become:

$$p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x}) = \prod_{i=1}^{n_\lambda} F_{\lambda,i}(B \cdot \underline{x})^{r_{\lambda,i}}.$$

Which is the decomposition of $p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x})$ into irreducible factors in $k(x_2, \ldots, x_n)$. Then

$$\sigma(f) \subset \sigma(f(B \cdot \underline{x})),$$

where $\sigma(f)$ is a subset of K, and $\sigma(f(B\cdot\underline{x}))$ is a subset of $k = \overline{K(x_1)}$. As n_λ is also the number of distinct irreducible factors of $p(B\cdot\underline{x})-\lambda q(B\cdot\underline{x})$ we get:

$$\rho(f) \leqslant \rho(f(B \cdot \underline{x})).$$

Now suppose that the result is true for $n-1$ variables. Then for $f(B \cdot \underline{x}) \in k(x_2, \ldots, x_n)$ we get:

$$\rho(f(B \cdot \underline{x})) < (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})).$$

Hence:

$$\begin{aligned}
\rho(f) &\leqslant \rho(f(B \cdot \underline{x})) \\
&< (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})) \\
&= (\deg_{(x_1,\ldots,x_n)} f)^2 + (\deg_{(x_1,\ldots,x_n)} f) \\
&= (\deg f)^2 + (\deg f)
\end{aligned}$$

□

If for $n = 2$ we start the induction with Lorenzini's bound $\rho(f) < (\deg f)^2$ we obtain with the same proof the following result for several variables, for K of any characteristic K and a better bound:

Theorem 5.3. *Let K be an algebraically closed field. Let $f \in K(\underline{x})$ be non-composite then $\rho(f) < (\deg f)^2$.*

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Irreducibility of hypersurfaces

avec Pierre Dèbes et Salah Najib

IRREDUCIBILITY OF HYPERSURFACES

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ABSTRACT. Given a polynomial P in several variables over an algebraically closed field, we show that except in some special cases that we fully describe, if one coefficient is allowed to vary, then the polynomial is irreducible for all but at most $\deg(P)^2 - 1$ values of the coefficient. We more generally handle the situation where several specified coefficients vary.

1. Introduction

Consider a polynomial in $n \geqslant 2$ variables over an algebraically closed field K. If it is reducible, it can be made irreducible by moving its coefficients (non-zero or not) away from some proper Zariski closed subset; polynomials in $n \geqslant 2$ variables are generically absolutely irreducible. This is no longer true if only one, specified, coefficient is allowed to vary. For example however one moves a non-zero coefficient of some homogeneous polynomial $P(x, y) \in K[x, y]$ of degree $d \geqslant 2$, it remains reducible. Yet it seems that this case is exceptional and that most polynomials are irreducible up to moving any fixed coefficient away from finitely many values. This paper is aimed at making this more precise.

1.1. **The problem.** The problem can be posed in general as follows: given an algebraically closed field K (of any characteristic) and a polynomial $P \in K[\underline{x}]$ (with $\underline{x} = (x_1, \dots, x_n)$), describe the "exceptional" *reducibility monomial sites* of P, that is those sets $\{Q_1, \dots, Q_\ell\}$ of monomials in $K[\underline{x}]$ for which $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is *generically reducible*, *i.e.* reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$[1], where $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ is a ℓ-tuple of independent indeterminates. When this is not the case, it follows from the Bertini-Noether theorem that the polynomial with shifted coefficients $P + \lambda_1^* Q_1 + \cdots + \lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$ for all $\underline{\lambda}^* = (\lambda_1^*, \dots, \lambda_\ell^*)$ in a non-empty Zariski open subset of K^ℓ (and the converse is true).

The situation $\ell = 1$ has been extensively studied in the literature, notably for $Q_1 = 1$, that is when it is the constant term that is moved: see works of Ruppert [Ru], Stein [St], Ploski [Pl], Cygan [Cy], Lorenzini [Lo], Vistoli [Vi], Najib [Na], Bodin [Bo] et al. The central result in this

[1] Given a field k, we denote by $\overline{k}$ an algebraic closure of k.

case, which is known as Stein's theorem, is that $P + \lambda$ is generically irreducible if and only if $P(\underline{x})$ is not a composed polynomial[2] (some say "indecomposable"); furthermore, the so-called spectrum of P consisting of all $\lambda^* \in K$ such that $P + \lambda^*$ is reducible in $K[\underline{x}]$, which from Bertini-Noether is finite in this case, is of cardinality $< \deg(P)$. This was first established by Stein in two variables and in characteristic 0, then extended to all characteristics by Lorenzini and finally generalized to n variables by Najib. The result also extends to arbitrary monomials Q_1, and in fact to arbitrary polynomials [Lo] [Bo]; the indecomposability assumption should be replaced by the condition that P/Q_1 is not a composed rational function, and the bound $\deg(P)$ by $\deg(P)^2$.

1.2. **Our results.** We fully describe the reducibility monomial sites of polynomials in the general situation $\ell \geqslant 1$ (theorem 3.3). We deduce simple practical criteria for generic irreducibility. These results can be combined with some ℓ-dimensional Stein-like description of the irreducibility set (proposition 4.1). The following three statements are typical illustrations of our study.

Recall K is an algebraically closed field of any characteristic. Below by Newton representation of a polynomial in n variables we merely mean the subset of all points $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that the monomial $x_1^{a_1} \cdots x_n^{a_n}$ appears in the polynomial with a non-zero coefficient.

Theorem 1.1. *Let $P(\underline{x}) \in K[\underline{x}]$ be a non constant polynomial and $Q(\underline{x})$ be a monomial of degree $\leqslant \deg(P)$ and relatively prime to P. Assume that the monomials of P together with Q do not lie on a line in their Newton representation*[3] *and that Q is not a pure power*[4] *in $K[\underline{x}]$. Then $P + \lambda Q$ is generically irreducible and the set of all $\lambda^* \in K$ such that $P + \lambda^* Q$ is reducible in $K[\underline{x}]$ is finite of cardinality $< \deg(P)^2$.*

Consequently a polynomial $P(x_1, \dots, x_n)$ can always be made irreducible by changing either the coefficient of x_1 or the coefficient of x_2, provided P is not divisible by $x_1 x_2$ (corollary 4.3).

The assumption on the monomials of P and Q in theorem 1.1 is there to avoid what we call the exceptional homogeneous case, that is, that P be of the form $h(m_1, m_2)$ with $h \in K[u, v]$ homogeneous and m_1, m_2 two monomials of degree $< \deg(P)$, in which case for any monomial $Q = m_1^k m_2^{d-k}$ $(0 \leqslant k \leqslant d = \deg(h))$, $P + \lambda Q$ is generically reducible.

[2] that is, is not of the form $r(S(\underline{x}))$ with $S \in K[\underline{x}]$ and $r \in K[t]$ with $\deg(r) \geqslant 2$.

[3] The result also holds if P is a monomial (in which case P and Q are lined up in the Newton representation).

[4] We say a polynomial $R \in K[\underline{x}]$ is a *pure power* if there exist $S \in K[\underline{x}]$ and $e > 1$ such that $R = S^e$. The monomial $Q(\underline{x}) = x_1^{e_1} \cdots x_n^{e_n}$ is not a pure power if and only if $e_1, \dots, e_n$ are relatively prime.

Pure power monomials Q, *e.g.* $Q = 1$, should also be excluded in theorem 1.1, but can nevertherless be dealt with under a slightly more general condition.

Theorem 1.2. *Let $P(\underline{x}) \in K[\underline{x}]$ be a non constant polynomial and $Q(\underline{x})$ be a monomial of degree $\leqslant \deg(P)$ and relatively prime to P. Assume P is not of the form $h(m, \psi)$ with $h \in K[u, v]$ an homogeneous polynomial, m a monomial dividing Q and $\psi \in K[\underline{x}]$ such that $\deg(P) > \max(\deg(m), \deg(\psi))$. Then $P+\lambda Q$ is generically irreducible and the set of all $\lambda^* \in K$ such that $P + \lambda^* Q$ is reducible in $K[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$.*

If P is of the excluded form then, for $Q = m^{\deg(h)}$, the polynomial $P + \lambda Q$ is generically reducible.

In the special case $Q = 1$, the assumption on P is that it is not of the form $h(1, \psi)$ with $h \in K[u, v]$ homogeneous, $\deg_v(h) \geqslant 2$ and $\psi \in K[\underline{x}]$: this corresponds to the classical hypothesis that P is not composed; theorem 1.2 is a generalization of Stein's theorem (except for the bound which can be taken to be $\deg(P)$ in this special case).

As another typical consequence of our approach, we obtain that for $\ell \geqslant 2$, reducibility monomials are even more rare.

Theorem 1.3. *Let $P \in K[\underline{x}]$ be a non constant polynomial and, for $\ell \geqslant 2$, $Q_1, \dots, Q_\ell$ be ℓ monomials of degree $\leqslant \deg(P)$ and such that $P, Q_1, \dots, Q_\ell$ are relatively prime. Assume the monomials of P together with $Q_1, \dots, Q_\ell$ do not lie on a line in their Newton representation. If $\mathrm{char}(K) = p > 0$ assume further that at least one of $P, Q_1, \dots, Q_\ell$ is not a p-th power. Then $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is generically irreducible and so $P + \lambda_1^* Q_1 + \cdots + \lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$ for all $(\lambda_1^*, \dots, \lambda_\ell^*)$ in a non-empty Zariski open subset of K^ℓ.*[5]

For example $P(x_1, \dots, x_n) + \lambda_1 x_1 + \cdots + \lambda_n x_n$ ($n \geqslant 2$) is generically irreducible. See corollary 4.3 for further related results.

1.3. **Organization of the paper.** A starting ingredient of our method is the Bertini-Krull theorem, which gives an iff condition for some polynomial $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ to be generically irreducible. The Bertini-Krull theorem is recalled in the preliminary section 2 which also introduces some basic definitions used in the rest of the paper.

Section 3 is the core of the paper. We investigate the Bertini-Krull conclusion in the specific context of our problem to finally obtain a general description of the reducibility monomial sites of a given polynomial (theorem 3.3).

[5]Prop. 4.1 gives a more explicit Stein-like description of the irreducibility set.

Section 4 is devoted to specializing the variables $\lambda_1, \ldots, \lambda_\ell$. For $\ell = 1$, we use the generalization of Stein's theorem due to Lorenzini [Lo] and Bodin [Bo] to give an upper bound for the number of exceptional values λ^* making $P + \lambda^* Q$ reducible in $K[\underline{x}]$. Similar estimates can be derived inductively for $\ell \geqslant 1$. We then complete the proof of the results from the introduction and give some further corollaries.

Finally in an appendix we prove a uniqueness result (theorem 5.2) in the Bertini-Krull theorem, which despite its basic nature only seemed to be available in some special cases.

1.4. **Main Data and Notation.** The following is given and will be retained throughout the paper:

- an algebraically closed field K of characteristic 0 or $p > 0$,
- an integer $\ell \geqslant 0$ and an ℓ-tuple $\underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ of independent variables (algebraically independent over K); for $\ell = 0$, the convention is that no variable is given,
- an integer $n \geqslant 2$ and an n-tuple $\underline{x} = (x_1, \ldots, x_n)$ of new independent variables (algebraically independent over $\overline{K(\underline{\lambda})}$),
- $\ell+1$ distinct (up to multiplicative constants) non-zero polynomials $P, Q_1, \ldots, Q_\ell \in K[\underline{x}]$ with $\max(\deg(P), \ldots, \deg(Q_\ell)) > 0$ and assumed further to be relatively prime if $\ell \geqslant 1$,
- $F(\underline{x}, \underline{\lambda}) = P(\underline{x}) + \lambda_1 Q_1(\underline{x}) + \cdots + \lambda_\ell Q_\ell(\underline{x})$, which is an irreducible polynomial in $K[\underline{x}, \underline{\lambda}]$ if $\ell \geqslant 1$. (For $\ell \geqslant 1$, $F(\underline{x}, \underline{\lambda})$ can be alternatively defined as a linear form in $(\lambda_0, \ldots, \lambda_\ell)$ (with $\lambda_0 = 1$) with distinct non-zero and relatively prime coefficients in $K[\underline{x}]$).

2. Bertini-Krull theorem and homogeneous decompositions

We first recall the Bertini-Krull theorem. We refer to [Sc, theorem 37] where equivalence between conditions (1) and (4) below is proved; equivalence between conditions (1), (2) and (3) is a special case of the standard Bertini-Noether theorem [FrJa, proposition 8.8].

Theorem 2.1 (Bertini, Krull)**.** *In addition to §1.4, assume $\ell \geqslant 1$. Then the following conditions are equivalent:*

(1) *$F(\underline{x}, \underline{\lambda}^*)$ is reducible in $K[\underline{x}]$ for all $\underline{\lambda}^* \in K^\ell$ such that* $\deg(F(\underline{x}, \underline{\lambda}^*)) = \deg_{\underline{x}}(F)$.
(2) *The set of $\underline{\lambda}^* \in K^\ell$ such that $F(\underline{x}, \underline{\lambda}^*)$ is reducible in $K[\underline{x}]$ is Zariski-dense.*
(3) *$F(\underline{x}, \underline{\lambda})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$.*
(4) (a) *either* $\operatorname{char} K = p > 0$ *and $F(\underline{x}, \underline{\lambda}) \in K[\underline{x}^p, \underline{\lambda}]$, where $\underline{x}^p = (x_1^p, \ldots, x_n^p)$,*

(b) *or there exist* $\phi, \psi \in K[\underline{x}]$ *with* $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$ *satisfying the following:*
(*) there is an integer $d > 1$ [6] *and* $\ell + 1$ *polynomials* $h_i(u, v) \in K[u, v]$ *homogeneous of degree* d *such that*

$$\begin{cases} P(\underline{x}) = h_0(\phi(\underline{x}), \psi(\underline{x})) = \sum_{k=0}^{d} a_{0k}\phi(\underline{x})^k \psi(\underline{x})^{d-k} \\ Q_1(\underline{x}) = h_1(\phi(\underline{x}), \psi(\underline{x})) = \dots \\ \vdots \\ Q_\ell(\underline{x}) = h_\ell(\phi(\underline{x}), \psi(\underline{x})) = \sum_{k=0}^{d} a_{\ell k}\phi(\underline{x})^k \psi(\underline{x})^{d-k} \end{cases}$$

which, setting $H(u, v, \underline{\lambda}) = h_0(u, v) + \sum_{i=1}^{\ell} \lambda_i h_i(u, v)$*, equivalently rewrites*

$$F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda}).$$

Remark 2.2. (1) In (4a), it follows from $F(\underline{x}, \underline{\lambda}) \in K[\underline{x}^p, \underline{\lambda}]$ that $P, Q_1, \dots, Q_\ell$ are in $K[x_1^p, \dots, x_n^p]$; as K is algebraically closed they are also p-th powers in $K[\underline{x}]$.
(2) It follows from the assumption "$P, Q_1, \dots, Q_\ell$ relatively prime" that the same is true for ϕ and ψ in (4b).

Definition 2.3. Given two polynomials $\phi, \psi \in K[\underline{x}]$ relatively prime and such that $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$,

(1) the polynomial F is said to be (ϕ, ψ)-*homogeneously composed (in degree d)* if there exists $H(u, v, \underline{\lambda}) \in \overline{K(\underline{\lambda})}[u, v]$ homogeneous (of degree d) in (u, v) such that $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$. The identity $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ is then called a (ϕ, ψ)-*homogeneous decomposition of* F. This definition is motivated by condition (4b) (*) of Bertini-Krull theorem.
(2) A (ϕ, ψ)-homogeneous decomposition $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ is said to be *maximal* if $\phi + \lambda\psi$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ [7].

Remark 2.4. (1) We also include in this definition the case $\ell = 0$ for which only the polynomial P is given. In this situation, the classical notion of composed polynomial corresponds to the special case of the "(ϕ, ψ)-homogeneously composed" property for which ϕ or ψ is constant.
(2) A natural question is whether a polynomial F can have several maximal homogeneous decompositions, *i.e.* relative to different couples (ϕ, ψ). In the appendix, we give a negative answer: for $\ell \geqslant 1$, the couple (ϕ, ψ) is unique, up to obvious transformations (theorem 5.2). A consequence is that the *maximality* condition

[6] This condition is actually a consequence of $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$.
[7] where λ is a new single variable (to be distinguished from the tuple $\underline{\lambda}$).

is equivalent (except in some special case) to the maximality of the degree of the homogeneous polynomial H (corollary 5.4), whence the terminology.

(3) From the Bertini-Krull theorem, "$\phi+\lambda\psi$ irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$" is equivalent to "$\phi+\lambda^*\psi$ irreducible in $K[\underline{x}]$ for at least one $\lambda^* \in K$ with $\deg(\phi+\lambda^*\psi) = \max(\deg(\phi), \deg(\psi))$" and also to "$\phi+\lambda^*\psi$ irreducible in $K[\underline{x}]$ for all but finitely many $\lambda^* \in K$".

The polynomial $F(x,y,\lambda) = x^4 - \lambda y^4$ admits the (x^2,y^2)-homogeneous decomposition $F(x,y,\lambda) = H_1(x^2,y^2,\lambda)$ with $H_1(u,v,\lambda) = u^2 - \lambda v^2$. It is not maximal as $x^2 - \lambda y^2 = (x-\sqrt{\lambda}y)(x+\sqrt{\lambda}y)$. This decomposition however can be refined to a (x,y)-homogeneous decomposition, which is maximal: namely we have $F(x,y,\lambda) = H_2(x,y,\lambda)$ with $H_2(u,v,\lambda) = u^4 - \lambda v^4$. This refinement is in fact always possible.

Proposition 2.5. *Assume $F(\underline{x},\underline{\lambda})$ is (ϕ_0,ψ_0)-homogeneously composed in degree d_0. Then there exists a maximal (ϕ,ψ)-homogeneous decomposition of F of degree $d \geqslant d_0$ and which is of degree $d > d_0$ if the initial decomposition is not maximal.*

Proof. Let $F(\underline{x},\underline{\lambda}) = H_0(\phi_0(\underline{x}),\psi_0(\underline{x}),\underline{\lambda})$ be a (ϕ_0,ψ_0)-homogeneous decomposition in degree d_0. If $\phi_0+\lambda\psi_0$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ then we are done. Otherwise apply the Bertini-Krull theorem to the polynomial $\phi_0+\lambda\psi_0$ (note that it is irreducible in $K[\lambda][\underline{x}]$ as ϕ_0 and ψ_0 are relatively prime) to conclude that there exist $\phi_1,\psi_1 \in K[\underline{x}]$ relatively prime and with $\max(\deg(\phi_0),\deg(\psi_0)) > \max(\deg(\phi_1),\deg(\psi_1))$ such that $\phi_0 + \lambda\psi_0$ is (ϕ_1,ψ_1)-homogeneously composed in degree $d_1 \geqslant 2$. Note that this conclusion also covers the extra possibility (4a) of theorem 2.1 in characteristic $p > 0$, which is here that $\phi_0 + \lambda\psi_0$ writes $\phi_1^p + \lambda\psi_1^p$ for some $\phi_1,\psi_1 \in K[\underline{x}]$. Straightforward calculations on homogeneous polynomials prove that F is then (ϕ_1,ψ_1)-homogeneously composed in degree $d_0d_1 > d_0$. We can iterate this process, which must stop because at each step the degree increases but remains $\leqslant \deg_{\underline{x}}(F)$. The last step yields a final homogeneous decomposition of F which is maximal. □

3. Reducibility monomial sites

We keep the notation of section 2 but assume in addition that $\ell \geqslant 1$ and that $Q_1,\ldots,Q_\ell$ are monomials such that $\deg(Q_i) \leqslant \deg(P)$, $i = 1,\ldots,\ell$. We set $Q_i = x_1^{e_{i1}} \cdots x_n^{e_{in}}$, $i = 1,\ldots,\ell$.

Definition 3.1. The set $\{Q_1,\ldots,Q_\ell\}$ is said to be a *reducibility monomial site* of P if $F(\underline{x},\underline{\lambda}) = P+\lambda_1Q_1+\cdots+\lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. If $\ell = 1$ we just say Q_1 is a *reducibility monomial.*

It is readily checked that any subset of a reducibility monomial site is a reducibility monomial site.

Definition 3.2. A polynomial $P \in K[\underline{x}]$ is said to be *homogeneous in two monomials* if P is (m_1, m_2)-homogeneously composed for some monomials m_1 and m_2 (which according to definition 2.3 should be relatively prime and such that $\deg(P) > \max(\deg(m_1), \deg(m_2))$).

This property can be easily detected thanks to the Newton representation of P (as already used in the introduction). Indeed, set $m_1 = x_1^{a_1} \cdots x_n^{a_n}$ and $m_2 = x_1^{b_1} \cdots x_n^{b_n}$. If P is homogeneous in m_1 and m_2, then P is a sum of monomials of the form:

$$m_1^k m_2^{d-k} = x_1^{db_1+k(a_1-b_1)} \cdots x_n^{db_n+k(a_n-b_n)} \qquad (k \in \{0, \ldots, d\})$$

The corresponding points $M_k = (db_1 + k(a_1 - b_1), \ldots, db_n + k(a_n - b_n))$ $(k = 0, \ldots, d)$ lie on a straight line in $\mathbb{Q}^n$.[8]

We will show below (theorem 3.3 (addendum 1)) that a (m_1, m_2)-homogeneous decomposition of P is maximal, that is $m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ if and only if m_1 and m_2 are not d-th powers in $K[\underline{x}]$ for some integer $d > 1$, or, equivalently, if $a_1, \ldots, a_n, b_1, \ldots, b_n$ are relatively prime.

3.1. **Main theorem.**

Our main result determines the reducibility monomial sites of a polynomial. We first state it in the general situation of a polynomial that is neither a monomial nor a pure power. The two remaining special cases are dealt with in two addenda. The proof is given in section 3.5.

Theorem 3.3 (general case)**.** *Assume $P(\underline{x})$ is not a monomial and is not a pure power in $K[\underline{x}]$.*

(1) *If P is homogeneous in two monomials, then given a maximal (m_1, m_2)-homogeneous decomposition $P = h(m_1, m_2)$ of degree δ with m_1 and m_2 monomials*[9]*, the reducibility monomial sites of P are all sets of monomials $m_1^k m_2^{\delta-k}$, $0 \leqslant k \leqslant \delta$, of degree $\leqslant \deg(P)$.*

[8] Note however that the monomials being lined up in the Newton representation is not sufficient for P to be homogeneous in two monomials: for example $P = xy + x^2y^4 + x^3y^7$ has that property but is not homogeneous in two monomials. It is of course easy to give a full test for some polynomial P to be homogeneous in terms of its Newton representation but writing out the exact condition is not very enlightening. See also remark 3.8.

[9] Such a decomposition exists (proposition 2.5) and is unique up to trivial transformations (lemma 3.7).

(2) *If P is not homogeneous in two monomials then the only possible reducibility monomial sites are singletons ($\ell = 1$) of the form $\{m^d\}$ with m a monomial relatively prime to P and $d \geqslant 2$. Furthermore the following should hold: $P = h(m, \psi)$ with $h \in K[u, v]$ homogeneous of degree d, $\psi \in K[\underline{x}]$ non monomial and* $\deg(P) > \max(\deg(m), \deg(\psi))$[10].

Remark 3.4. (1) In the homogeneous case (1), the reducibility monomials $m_1^k m_2^{\delta-k}$ also are on the line formed by the monomials of P in its Newton representation.

(2) In case (2) we do not know whether there may be several reducibility monomials of the form m^d. This is related to the possibility that P can be written $P = h(m, \psi)$ as in the statement in several different ways. In the appendix we give a example of a polynomial P with several such decompositions (example 5.3). However the two monomials m^d associated to the two homogeneous decompositions of P shown there are x^2 and y^2; the second one is not relatively prime to P and so is not a reducibility monomial according to our definitions.

(3) In case (2) where $P = h(m, \psi)$, by setting $g(t) = h(1, t)$ we obtain $P/m^d = g(\psi/m)$ is a *composite rational function* as considered in [Bo] (of special form though as g is here a polynomial).

3.2. **The monomial case.** Here we consider the case P is a monomial $\gamma\, x_1^{e_1} \cdots x_n^{e_n}$ (with $\gamma \in K, \gamma \neq 0$). The argument below can be viewed as an easy special case of the general method.

From §2, if $F(\underline{x}, \underline{\lambda})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$, then equivalently either $F(\underline{x}, \underline{\lambda}) \in K[\underline{x}^p, \underline{\lambda}]$ (with $\mathrm{char}(K) = p > 0$) or $F(\underline{x}, \underline{\lambda})$ is (ϕ, ψ)-homogeneously composed in degree d for some $\phi, \psi \in K[\underline{x}]$. In the latter case, factor the homogeneous polynomials involved in the decomposition as products of linear forms to obtain

$$\begin{cases} P(\underline{x}) = \prod_{k=1}^{\mu_0} (\alpha_{0k}\phi(\underline{x}) + \beta_{0k}\psi(\underline{x}))^{r_{0k}} \\ Q_i(\underline{x}) = \prod_{k=1}^{\mu_i} (\alpha_{ik}\phi(\underline{x}) + \beta_{ik}\psi(\underline{x}))^{r_{ik}} \quad (i = 1, \ldots, \ell) \end{cases}$$

where the $(\alpha_{ik}, \beta_{ik})$ are non-zero, pairwise non proportional and the integers r_{ik} are > 0 and satisfy $\sum_{k=1}^{\mu_i} r_{ik} = d$ $(i = 0, \ldots, \ell)$.

All the factors appearing in the right-hand side terms are necessarily monomials and at least two of them are non proportional (as $P, Q_1, \ldots, Q_\ell$ are relatively prime). Therefore up to changing (ϕ, ψ) to $L(\phi, \psi)$ for some $L \in \mathrm{GL}_2(K)$ one may assume that ϕ and ψ themselves are two monomials m_1 and m_2. Taking into account that $P, Q_1, \ldots, Q_\ell$

[10]By proposition 2.5 we may also impose that $\psi + \lambda m$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$.

are monomials and that they are relatively prime, we obtain the following characterization (the converse is clear).

Theorem 3.3 (addendum 1)**.** *If P is a monomial the following are equivalent:*

(1) *The polynomial $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ (that is, $\{Q_1, \ldots, Q_\ell\}$ is a reducibility monomial site of P),*

(2) (a) *either* $\operatorname{char} K = p > 0$ *and $P, Q_1, \ldots, Q_\ell \in K[\underline{x}^p]$,*

(b) *or $P, Q_1, \ldots, Q_\ell$ are of the form $m_1^k m_2^{d-k}$ $(0 \leqslant k \leqslant d)$ for some relatively prime monomials m_1 and m_2 and some integer $d > 1$, and they include m_1^d and m_2^d.*
Furthermore, for all (ϕ, ψ)-homogeneous decompositions of $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$, (ϕ, ψ) is a couple of monomials, up to some element $L \in \mathrm{GL}_2(K)$.

Remark 3.5. In general there may be several couples (m_1, m_2) such that P is of the form $m_1^k m_2^{d-k}$, and so several corresponding reducibility sites for P. For example $P = x^3 y^2$ is homogeneously composed for both couples of monomials (x^3, y^2) and $(x^3 y^2, 1)$ and both decompositions are maximal. In the non monomial case, this will not happen: up to trivial transformations the couple (m_1, m_2) is uniquely determined by P (see lemma 3.7).

3.3. **Pure power case.** In the case P is a pure power in $K[\underline{x}]$, the three following possibilities can occur:

(1) P is homogeneous in two monomials. In this case let $P = h(m_1, m_2)$ be a maximal homogeneous decomposition of degree δ in two monomials m_1 and m_2 and set $\mathcal{M}_1 = \{m_1^k m_2^{\delta-k} \mid 0 \leqslant k \leqslant \delta\}$. All subsets of $\mathcal{M}_1$ are reducible monomial sites.

(2) P admits a maximal (m, ψ)-homogeneous decomposition in degree d, with m a monomial and $\psi \in K[\underline{x}]$ non monomial. In this case, if $\deg(m^d) \leqslant \deg(P)$, then m^d is a reducibility monomial.

(3) $\operatorname{char}(K) = p > 0$ and $P \in K[\underline{x}^p]$. In this case set $\mathcal{M}_3 = \{m^p \mid m \text{ is a monomial and} \deg(m^p) \leqslant \deg(P)\}$. All subsets of $\mathcal{M}_3$ are reducible monomial sites.

Theorem 3.3 (addendum 2)**.** *Assume P is a pure power but is not a monomial. Then the reducibility monomial sites of P are those described in possibilities (1), (2) and (3).*

The following observations make the pure power case rather special:

(a) possibility (2) is always satisfied: indeed by assumption we have $P = S^e$ for some $S \in K[\underline{x}]$ and some integer $e > 1$, which is a (m, S)-homogeneous decomposition of degree e for any monomial m relatively prime to S; the corresponding monomials m^e with $\deg(m^e) \leqslant \deg(P)$ are reducibility monomials. However there may be other kinds of decompositions $P = h(m, \psi)$. For example, take $P(x, y) = (2y^3 - x^4)^2 x^4$. Squares monomials of degree $\leqslant 12$ are reducibility monomials. Now for $m = y^3$, $\psi = y^3 - x^4$ and $h(u, v) = (u + v)^2(u - v)$, we also have $P = h(m, \psi)$ and so $m^3 = y^9$ is another reducibility monomial of P.

(b) possibilities (1), (2) and (3) can occur simultaneously. Take for example $P(x, y) = (x^2 - y^3)^3$. Then P is homogeneous in the two monomials x^2 and y^3; the corresponding set $\mathcal{M}_1$ is $\mathcal{M}_1 = \{x^6, x^4y^3, x^2y^6, y^9\}$. As P is a third power, each of the monomials 1, x^3, y^3, x^6, x^3y^3, y^6, x^9, x^6y^3, x^3y^6, y^9 is a reducibility monomial. Finally if $\mathrm{char}(K) = 3$, then every subset of $\mathcal{M}_3 = \{1, x^3, y^3, x^6, x^3y^3, y^6, x^9, x^6y^3, x^3y^6, y^9\}$ is a reducibility monomial site.

3.4. **Lemmas.** The two following lemmas will be used in the proof of theorem 3.3.

Lemma 3.6. *Given two monomials $m_1, m_2 \in K[\underline{x}]$ such that we have* $\max(\deg(m_1), \deg(m_2)) > 0$*, the following are equivalent:*

(i) *there exists $\lambda^* \in K$, $\lambda^* \neq 0$, such that $m_1 + \lambda^* m_2$ is irreducible in $K[\underline{x}]$,*
(ii) *for all $\lambda^* \in K$, $\lambda^* \neq 0$, $m_1 + \lambda^* m_2$ is irreducible in $K[\underline{x}]$,*
(iii) *$m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$.*

Proof. The equivalence (iii)$\Leftrightarrow$(i) is a special case of the Bertini-Krull theorem and (ii)$\Rightarrow$(i) is trivial. We are left with proving (i)$\Rightarrow$(ii). Assume there exist $\lambda_1^*, \lambda_2^* \in K$, both non zero and such that $m_1 + \lambda_1^* m_2$ is reducible and $m_1 + \lambda_2^* m_2$ is irreducible in $K[\underline{x}]$.

Set $m_1 = x_1^{a_1} \cdots x_n^{a_n}$ and $m_2 = x_1^{b_1} \cdots x_n^{b_n}$. One may assume that $\deg(m_2) > 0$ and so for example $b_1 > 0$. If $a_1 > 0$ then x_1 divides $m_1 + \lambda_2^* m_2$ and so $m_1 = m_2 = x_1$ (up to some non-zero multiplicative constants) in which case the result is obvious. Thus one may assume $a_1 = 0$. If $m_1(\underline{x}) + \lambda_1^* m_2(\underline{x}) = R(\underline{x}) \cdot S(\underline{x})$ is a non trivial factorization of $m_1 + \lambda_1^* m_2$ ($\deg(R), \deg(S) > 0$), we have

$$m_1 + \lambda_2^* m_2 = R\left((\lambda_1^{*-1}\lambda_2^*)^{\frac{1}{b_1}} x_1, x_2, \ldots, x_n\right) \cdot S\left((\lambda_1^{*-1}\lambda_2^*)^{\frac{1}{b_1}} x_1, x_2, \ldots, x_n\right)$$

which contradicts the irreducibility of $m_1 + \lambda_2^* m_2$. □

Lemma 3.7. *Assume $P(\underline{x})$ is not a monomial and is given with a maximal (m_1, m_2)-homogeneous decomposition $P = h(m_1, m_2)$ of degree d with m_1 and m_2 monomials.*

(1) *If $P = h'(m_1', m_2')$ is another maximal homogeneous decomposition of degree d' of P in monomials m_1' and m_2', then either*

($m_1 = am'_1$ and $m_2 = bm'_2$) or ($m_1 = am'_2$ and $m_2 = bm'_1$), for some non-zero constants $a, b \in K$, and $d = d'$.

(2) *There is no maximal homogeneous (m, ψ)-decomposition of $P = h'(m, \psi)$ with $\psi \in K[\underline{x}]$ non monomial and m a monomial relatively prime to P and not a monomial of ψ unless $P = \psi^{d''}$ with ψ homogeneous in m_1 and m_2 and $d'' \geqslant 2$.*

Proof. We can write

$$P = h(m_1, m_2) = \prod_{k=1}^{\mu} (\alpha_k m_1 + \beta_k m_2)^{r_k} \tag{$**$}$$

where the (α_k, β_k) are non-zero and pairwise non-proportional and the integers r_k are > 0 and satisfy $\sum_{k=1}^{\mu} r_k = d$.

(1) As P is not a monomial there exists $k \in \{1, \dots, \mu\}$ such that $\alpha_k \beta_k \neq 0$. Then by lemma 3.6 $\alpha_k m_1 + \beta_k m_2$ is irreducible in $K[\underline{x}]$.

Assume P has another maximal homogeneous decomposition in monomials m'_1 and m'_2

$$P = h'(m'_1, m'_2) = \prod_{k=1}^{\mu'} (\alpha'_k m'_1 + \beta'_k m'_2)^{r'_k}$$

where the (α'_k, β'_k) are non-zero, pairwise non proportional and the integers r'_k are > 0 and satisfy $\sum_{k=1}^{\mu'} r'_k > 1$. From the unique factorization property in the domain $K[\underline{x}]$, there exists $h \in \{1, \dots, \mu'\}$ with $\alpha'_h \beta'_h \neq 0$ such that, up to a non-zero multiplicative constant, we have $\alpha_k m_1 + \beta_k m_2 = \alpha'_h m'_1 + \beta'_h m'_2$. As m_1, m_2, m'_1, m'_2 are monomials we obtain the desired conclusion.

Remark 3.8. In fact the monomials m_1 and m_2 of some maximal homogeneous decomposition of P can be easily recovered from the Newton representation of P. Indeed, using the notation from the beginning of section 3, for any two distinct points M_h and M_k, we have $\overrightarrow{M_k M_h} = (k-h)\overrightarrow{\Delta}$ where $\overrightarrow{\Delta} = (a_1 - b_1, \dots, a_n - b_n)$. As $\min(a_j, b_j) = 0$, $j = 1, \dots, \ell$, the non-zero exponents of m_1 (resp. of m_2) correspond to the positive components (resp. to the negative components) of $\overrightarrow{\Delta}$. As $a_1, \dots, a_n, b_1, \dots, b_n$ are relatively prime, these exponents correspond to the components of $\overrightarrow{M_k M_h}$ divided by their g.c.d.

(2) Suppose P has a maximal (m, ψ)-homogeneous decomposition (with m and ψ as in the statement)

$$P = h'(m,\psi) = \prod_{k=1}^{\mu'} (\alpha'_k \psi + \beta'_k m)^{r'_k}$$

where the (α'_k, β'_k) are non-zero and pairwise non-proportional and the integers r'_k are > 0 and satisfy $\sum_{k=1}^{\mu'} r'_k = d' > 1$.

Consider first the case there exists $h \in \{1,\dots,\mu'\}$ with $\alpha'_h\beta'_h \neq 0$. Comparing with $(**)$ above we obtain that the polynomial $\alpha'_h\psi + \beta'_h m$ is a product of say ν irreducible factors $\alpha_k m_1 + \beta_k m_2$ with $\alpha_k\beta_k \neq 0$ (irreducible by lemma 3.6) and possibly some monomial ρ. As $\alpha'_h\psi + \beta'_h m$ has at least 3 monomials, the integer ν is $\geqslant 2$. Thus $\alpha'_h\psi+\beta'_h m$ can be written $\rho\,\kappa(m_1,m_2)$ with $\kappa \in K[u,v]$ homogeneous of degree $\nu \geqslant 2$. As m is not a monomial of ψ, conclude that, up to non zero constants in K, m is one of the monomials of $\kappa(m_1,m_2)$ multiplied by ρ and that ψ is the sum of the other monomials of $\kappa(m_1,m_2)$, also multiplied by ρ. Now as ψ and m are relatively prime, ρ is a non-zero constant in K. But then $m+\lambda\psi$ is (m_1,m_2)-homogeneously composed in degree ν, which contradicts the maximality of the (m,ψ)-decomposition.

Assume next that $\alpha'_h\beta'_h = 0$ for all $h = 1,\dots,\mu'$. If no coefficient α'_h is zero, then $P = \psi^{d'}$ (up to some non-zero multiplicative constant). If some coefficient α'_h is zero, then m divides P and as P and m are assumed to be relatively prime, m is a non-zero constant in K. Conclude in both cases that $P = \psi^{d''}$ (up to some non-zero multiplicative constant) where d'' is the number of coefficients α'_h that are non-zero (counted with the multiplicities r'_k); we have $d'' \leqslant d'$ and $d'' \geqslant 2$ for otherwise we would have $\deg(P) \leqslant \max(\deg(\psi),\deg(m))$. Observe next that the exponents r_k are all divisible by d'': if $\alpha_k\beta_k \neq 0$, this is because $\alpha_k m_1 + \beta_k m_2$ is irreducible in $K[\underline{x}]$ and for the possible two factors that are powers of m_1 and m_2, because m_1 and m_2 are relatively prime. Conclude ψ is as announced homogeneous in m_1 and m_2. □

3.5. **Proof of theorem 3.3.** Addendum 1 has already been proved (in section 3.2) so we may assume P is not a monomial.

3.5.1. *Preliminary discussion:* Let $\{Q_1,\dots,Q_\ell\}$ ($\ell \geqslant 1$) be a reducibility monomial site of P.

From remark 2.2 the case (4a) in the Bertini-Krull theorem can only occur if P is a pure power, and in this case the conclusion corresponds to possibility (3) of theorem 3.3 (addendum 2).

Suppose now it is part (4b) of the Bertini-Krull theorem that holds. That is, the polynomial $F(\underline{x},\underline{\lambda}) = P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ has a (ϕ,ψ)-homogeneous decomposition in degree d for some $\phi,\psi \in K[\underline{x}]$, which in addition we may and will assume to be maximal (proposition 2.5).

Thus we have $P(\underline{x}) = h_0(\phi(\underline{x}), \psi(\underline{x}))$ and $Q_i(\underline{x}) = h_i(\phi(\underline{x}), \psi(\underline{x}))$ $(i = 1, \dots, \ell)$ for some homogeneous polynomials $h_0, \dots, h_\ell \in K[u, v]$ of degree d. Note that as $\deg(P) \geqslant \deg(Q_i)$, $i = 1, \dots, \ell$, we have $\deg_{\underline{x}}(F) = \deg(P) > \max(\deg(\phi), \deg(\psi))$ and so $P = h_0(\phi, \psi)$ is still a (ϕ, ψ)-homogeneous decomposition of P. Write then $h_i(u, v) = \prod_{k=1}^{\mu_i}(\alpha_{ik}u + \beta_{ik}v)^{r_{ik}}$ with, for each $i = 0, \dots, \ell$, the $(\alpha_{ik}, \beta_{ik})$ non-zero and pairwise non proportional and the integers $r_{ik} > 0$ and satisfying $\sum_{k=1}^{\mu_i} r_{ik} = d$. Unless $\ell = 1$ and Q_1 is constant, one may assume Q_1 is a non constant monomial and then all factors $\alpha_{1k}\phi(\underline{x}) + \beta_{1k}\psi(\underline{x})$ $(k = 1, \dots, \mu_1)$ are monomials and at least one, say m, is non constant. If $\ell = 1$ and Q_1 is constant, then ϕ or ψ, say ϕ is constant. In all cases, up to changing (ϕ, ψ) to $L(\phi, \psi)$ for some $L \in \mathrm{GL}_2(K)$, one may assume that ϕ is a monomial m and that m is not a monomial of ψ. Observe then that if ψ has at least two monomials then $Q_i = h_i(m, \psi)$ can be a monomial only if $h_i(u, v) = u^d$ and so $\ell = 1$ and $Q_1 = m^d$.

3.5.2. *1st case:* P is homogeneous in two monomials.

Let $P = h(m_1, m_2)$ be a maximal (m_1, m_2)-homogeneous decomposition in degree δ with m_1 and m_2 monomials. From above $P = h_0(m, \psi)$ is another maximal homogeneous decomposition.

If ψ itself is a monomial then from lemma 3.7 (1), we have $d = \delta$ and $(m, \psi) = (am_1, bm_2)$ or $(m, \psi) = (bm_2, am_1)$ for some non-zero constants $a, b \in K$. Conclude each Q_i is homogeneous in m_1 and m_2 in degree δ and as Q_i is a monomial, it should be of the form $m_1^k m_2^{\delta-k}$ for some $k \in \{0, \dots, \delta\}$. Conversely, any set consisting of such monomials is clearly a reducibility monomial site of P.

Assume next that ψ is not a monomial. From the preliminary discussion $\ell = 1$ and $Q_1 = m^d$. In particular, P and m are relatively prime. It follows from lemma 3.7 (2) that $P = \psi^{d''}$ with ψ homogeneous in m_1 and m_2 and $d'' \geqslant 2$. In particular this can only occur if P is a pure power. Thus we are done with case (1) of theorem 3.3 (general) where P being a pure power is excluded. If P is a pure power, what we have obtained is contained in possibilities (1) and (2) from theorem 3.3 (addendum 2).

3.5.3. *2nd case:* P is not homogeneous in two monomials.

In this case ψ is not a monomial and the desired conclusions — that is, on one hand, case (2) of theorem 3.3 (general) and on the other hand that only possibility (2) can occur apart from possibilities (1) and (3) in theorem 3.3 (addendum 2) — are part of the preliminary discussion.

4. Specialization

In this section we explain how irreducibility properties of $F(\underline{x}, \underline{\lambda})$ can be preserved by specialization of the variables λ_i in K. This is the last stage towards the results stated in the introduction.

4.1. **Using Stein like results.**

Proposition 4.1. *Assume $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ (that is $\{Q_1, \ldots, Q_\ell\}$ is not a reducibility monomial site of P). Then for every $i = 1, \ldots, \ell$, the set of $\lambda_i^* \in K$ such that $P + \lambda_1 Q_1 + \cdots + \lambda_{i-1} Q_{i-1} + \lambda_i^* Q_i + \lambda_{i+1} Q_{i+1} + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_\ell)}[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$.*

Consequently, for every $\lambda_1^* \in K$ but in a finite set of cardinality $< \deg(P)^2$, for every $\lambda_2^* \in K$ but in a finite set of cardinality $< \deg(P)^2$ (depending on λ_1^*),..., for every $\lambda_\ell^* \in K$ but in a finite set of cardinality $< \deg(P)^2$ (depending on $\lambda_1^*, \ldots, \lambda_{\ell-1}^*$), the polynomial $P + \lambda_1^* Q_1 + \cdots + \lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$.

Remark 4.2. The assumption "$P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$" holds if it holds for a smaller ℓ, in particular if P itself is irreducible in $K[\underline{x}]$. This follows immediately from the equivalence of (1) and (3) in the Bertini-Krull theorem.

Proof of proposition 4.1. With no loss of generality we may assume $i = 1$ in the first part. Set $G = P + \lambda_2 Q_2 + \cdots + \lambda_\ell Q_\ell$ and $L = \overline{K(\lambda_2, \ldots, \lambda_\ell)}$. By hypothesis, $G + \lambda_1 Q_1$ is irreducible in $\overline{L(\lambda_1)}[\underline{x}]$. From the generalization of Stein's theorem to general pencils of hypersurfaces $P + \lambda Q$ (and not just the curves $P + \lambda$) given in [Bo] (relying on [Ru], [Lo] and [Na]), the set of $\lambda^* \in L$ such that $G + \lambda^* Q_1$ is reducible in $L[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$. The second part is an easy induction. □

4.2. **Proof of the results from the introduction.**

4.2.1. *Proof of theorem 1.1.* Due to the assumptions on the monomials of P and Q, Q cannot be a reducibility monomial in the homogeneous case (1) from theorem 3.3 (general) nor in possibility (1) from theorem 3.3 (addendum 2). The monomial Q not being a pure power forbids condition (2) from theorem 3.3 (addendum 1) (with $\ell = 1$ and $Q_1 = Q$) to happen and Q to be a reducibility monomial in case (2) from theorem 3.3 (general) and in possibilities (2) and (3) from theorem 3.3 (addendum 2). Therefore $P + \lambda Q$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$. Apply then proposition 4.1 to complete the proof of theorem 1.1.

4.2.2. *Proof of theorem 1.2.* Assume as in theorem 1.2 that P is not of the form $h(m, \psi)$ with $h \in K[u, v]$ homogeneous of degree $\geqslant 2$, $\psi \in K[\underline{x}]$ and m a monomial dividing Q. In particular P is not a pure power (for otherwise P is of this form with $h(u, v) = v^d$ for some $d > 1$ and $m = 1$). We show below that assuming Q is a reducibility monomial of P leads to a contradiction.

The homogeneous case (1) from theorem 3.3 (general) can be ruled out as follows. If this case occured, then by assumption neither m_1 nor m_2 could divide Q but this is not possible in view of the form of the reducibility monomial sites in this case.

The case P is a monomial can also be excluded: condition (2) from theorem 3.3 (addendum 1) (with $\ell = 1$ and $Q_1 = Q$) cannot hold since P is not a pure power.

The remaining possibility (2) from theorem 3.3 (general) cannot happen either since in this case P should be of the form $h(m, \psi)$ as above and $Q = m^d$ (and so m divides Q).

Conclude Q is not a reducibility monomial of P, that is, $P + \lambda Q$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$, and apply proposition 4.1 to complete the proof of theorem 1.2.

4.2.3. *Proof of theorem 1.3.* Here $\ell \geqslant 2$. The reducibility monomial sites of cardinality ℓ can only occur in the homogeneous cases from theorem 3.3 or in characteristic $p > 0$. But these possibilities are ruled out by the assumptions. Therefore $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. Apply then the classical Bertini-Noether theorem [FrJa, proposition 8.8] or alternatively proposition 4.1 to conclude the proof.

4.3. **Examples.** The following variations around Stein's theorem illustrate our main results. We leave the reader find others.

Corollary 4.3. *Let $P \in K[x_1, \ldots, x_n]$ be a polynomial in $n \geqslant 2$ variables and with coefficients in the algebraically closed field K. In each of the following situations it can be concluded that $P + \lambda^* Q$ is irreducible for all but at most* $\deg(P)^2 - 1$ *values of $\lambda^* \in K$.*

1. *$P \notin K[x_1]$, P is not divisible by x_1 and $Q = x_1$.*
2. *P is not divisible by $x_1 x_2$ and $Q = x_i$ for some $i \in \{1, 2\}$.*
3. *$n = 2$, $P(x, y) \in K[x, y]$ is homogeneous of degree $d > 1$ but is not a pure power and $Q = x^i y^j$ is a monomial of degree $i + j < d$ and relatively prime to P.*

Proof. (1) Suppose that $P(x_1, \ldots, x_n) + \lambda x_1$ is reducible in $\overline{K(\lambda)}[\underline{x}]$. As x_1 is not a pure power, it follows from theorem 3.3 that $P = h(m_1, m_2)$ for some homogeneous polynomial $h \in K[u, v]$ of degree $d > 1$ and some monomials m_1 and m_2 and that $x_1 = m_1^k m_2^{d-k}$ for some $k \in \{0, \dots, d\}$. Then we have necessarily $\{m_1, m_2\} = \{1, x_1\}$. But then $P = h(m_1, m_2)$ contradicts the assumption $P \notin K[x_1]$. Thus $P(x_1, \dots, x_n) + \lambda x_1$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ and the result follows from proposition 4.1.

(2) One may assume P is not constant and that x_1 does not divide P. If $P \notin K[x_1]$, the result follows from (1). If $P \in K[x_1]$, then x_2 does not divide P and $P \notin K[x_2]$, so (1) applies to conclude the proof.[11]

(3) Irreducibility of $P(x,y) + \lambda x^i y^j$ in $\overline{K(\lambda)}[x,y]$ readily follows from theorem 3.3 (general & addendum 1): just note P is homogeneous in the two monomials $m_1 = x$ and $m_2 = y$, which are relatively prime, of degree $< \deg(P)$ and such that $m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[x,y]$. Apply then proposition 4.1 to complete the proof. □

5. Appendix: Uniqueness in the Bertini-Krull theorem

The goal of this appendix is Theorem 5.2 below. We come back to the general notation from subsection 1.4; $Q_1, \dots, Q_\ell$ are not necessarily monomials as in sections 4 and 3. In addition we assume $\ell \geqslant 1$.

We need a preliminary adjustment of definition 2.3. Given a (ϕ, ψ)-homogeneous decomposition $F(\underline{x}, \underline{\lambda}) = H(\phi, \psi, \underline{\lambda})$, assume there exists $(\alpha, \beta) \neq (0,0)$ in K^2 such that $\alpha\phi + \beta\psi$ is constant in $\underline{x}$ (that is, is in K). Then multiplying $H(u, v, \underline{\lambda})$ by any power $(\alpha u + \beta v)^e$ yields another decomposition $F(\underline{x}, \underline{\lambda}) = \tilde{H}(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ with $\tilde{H}$ homogeneous (in u, v) of degree $\tilde{d} = d + e$. Conversely if $H(u, v, \underline{\lambda})$ has linear factors $\alpha u + \beta v$ (in $\overline{K(\underline{\lambda})}[u, v]$) with $\alpha\phi + \beta\psi$ constant in $\underline{x}$, then they are all equal, up to some constant in $\overline{K(\underline{\lambda})}$, to a same linear form $\alpha_0 u + \beta_0 v \in K[u, v]$ and the homogeneous polynomial $H'(u, v, \underline{\lambda})$ obtained from $H(u, v, \lambda)$ by dividing by all possible such factors $\alpha u + \beta v$ still induces a decomposition $F(\underline{x}, \underline{\lambda}) = H'(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ with H' homogeneous of degree $d' \leqslant d$. Note we still have $d' \geqslant 2$ as $d' \leqslant 1$ contradicts $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$.

Definition 5.1. Given two polynomials $\phi, \psi \in K[\underline{x}]$ relatively prime with $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$, a (ϕ, ψ)-homogeneous decomposition $F = H(\phi, \psi, \underline{\lambda})$ is said to be *reduced* if the polynomial H has no linear factor $\alpha u + \beta v \in K[u, v]$ such that $\alpha\phi + \beta\psi$ is constant in $\underline{x}$.

[11] Statement 2 can be alternatively deduced from theorem 1.1. Indeed if x_1 does not divide P, there is some point $(0, a_2, \dots, a_n)$ on the Newton representation of P. Take $Q = x_1$. Then Q and P are relatively prime and x_1 is not a pure power. Theorem 3.3 (addendum 1) handles the case P is a monomial. Otherwise P has another non-zero monomial but the corresponding coefficient cannot be on the line from $(0, a_2, \dots, a_n)$ to $(1, 0, \dots, 0)$ (which corresponds to x_1).

From above a reduced (ϕ,ψ)-homogeneous decomposition of F is easily obtained from any (ϕ,ψ)-homogeneous decomposition of F.

Also note that if there exists $(\alpha,\beta)\neq(0,0)$ in K^2 such that $\alpha\phi+\beta\psi$ is constant, then up to applying some linear transformation $L\in \mathrm{GL}_2(K)$ to (ϕ,ψ), one may assume $\phi=1$ and so this can only happen if F is a composed polynomial (over $\overline{K(\underline{\lambda})}$). Thus only in this case does definition 5.1 add something to definition 2.3.

Theorem 5.2. *Assume $\ell\geqslant 1$. If $F(\underline{x},\underline{\lambda})=P(\underline{x})+\lambda_1Q_1(\underline{x})+\cdots+\lambda_\ell Q_\ell(\underline{x})$ admits two maximal homogeneous decompositions:*

$$F(\underline{x},\underline{\lambda})=H_1(\phi_1(\underline{x}),\psi_1(\underline{x}),\underline{\lambda})=H_2(\phi_2(\underline{x}),\psi_2(\underline{x}),\underline{\lambda})$$

then there exists $L\in\mathrm{GL}_2(K)$ such that $(\phi_1,\psi_1)=L(\phi_2,\psi_2)$. Furthermore if the two decompositions are reduced then we have $c\cdot H_2(u,v,\underline{\lambda})=H_1(u,v,\underline{\lambda})\circ L(u,v)$ for some constant $c\in K$.

Example 5.3. Theorem 5.2 does not extend to the case $\ell=0$. Here is a counter-example. Let $P(x,y)=y(x+y)(y^2+xy-2x)$. We have the two maximal homogeneous decompositions:
- $P=h_1(\phi_1,\psi_1)$ with $h_1(u,v)=v^2-u^2$, $\phi_1=x$, $\psi_1=(y-1)(x+y)+y$,
- $P=h_2(\phi_2,\psi_2)$ with $h_2(u,v)=uv$, $\phi_2=y$, $\psi_2=(x+y)(y^2+xy-2x)$.
These two decompositions are distinct even up to elements of $\mathrm{GL}_2(K)$.

Corollary 5.4. *All reduced maximal homogeneous decompositions of F are of the same degree, say δ. Furthermore if F is not a composed polynomial over $\overline{K(\underline{\lambda})}$, any homogeneous decomposition of F is of degree $\leqslant\delta$ and equality holds if and only if it is maximal.*

Proof of theorem 5.2. Consider a reduced maximal homogeneous decomposition $F(\underline{x},\underline{\lambda})=H(\phi(\underline{x}),\psi(\underline{x}),\underline{\lambda})$. Write the homogeneous polynomial $H(u,v,\underline{\lambda})$ (in u,v) as a product $\prod_{i=1}^d(\alpha_i(\underline{\lambda})u+\beta_i(\underline{\lambda})v)$ of linear forms in u,v with coefficients in $\overline{K(\underline{\lambda})}$. Thus we have

$$P(\underline{x})+\sum_{i=1}^{\ell}\lambda_iQ_i(\underline{x})=\prod_{k=1}^{d}(\alpha_k(\underline{\lambda})\phi(\underline{x})+\beta_k(\underline{\lambda})\psi(\underline{x})).$$

The result will be easily deduced from these two claims and the unique factorization property in the domain $\overline{K(\underline{\lambda})}[\underline{x}]$.

(a) There are at least two factors $\alpha_k(\underline{\lambda})\phi(\underline{x})+\beta_k(\underline{\lambda})\psi(\underline{x})$ that are non constant in $\underline{x}$ and non proportional (by some constant in $\overline{K(\underline{\lambda})}$).
(b) All factors $\alpha_k(\underline{\lambda})\phi(\underline{x})+\beta_k(\underline{\lambda})\psi(\underline{x})$ $(k=1,\dots,d)$ are irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ and are not in $K[\underline{x}]$ (even up to constants in $\overline{K(\underline{\lambda})}$).

Proof of claim (a). First note that due to definition 5.1, no factor $\alpha_k\phi+\beta_k\psi$ is in $\overline{K(\underline{\lambda})}$. Assume (a) does not hold. Then $F(\underline{x},\underline{\lambda})$ is of the form αM^d with $\alpha \in \overline{K(\underline{\lambda})}$ and $M = \alpha_1\phi + \beta_1\psi$. Taking the derivative with respect to λ_i shows that M^{d-1} divides Q_i in $\overline{K(\underline{\lambda})}[\underline{x}]$, $i = 1,\dots,\ell$. But as M^d divides $F(\underline{x},\underline{\lambda})$, we obtain that M^{d-1} divides P as well. A contradiction as $\deg(M) > 0$ and $P, Q_1,\dots,Q_\ell$ are assumed to be relatively prime.

Proof of claim (b). Assume that for some $k \in \{1,\dots,d\}$, $\alpha_k\phi + \beta_k\psi$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. One may assume that $\deg(\psi) > 0$ and $\beta_k \neq 0$. If $\alpha_k \neq 0$, set $\mu(\underline{\lambda}) = \beta_k(\underline{\lambda})/\alpha_k(\underline{\lambda})$. The polynomial $\phi(\underline{x}) + \mu(\underline{\lambda})\psi(\underline{x})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ and consequently so are the polynomials $\phi(\underline{x}) + \mu(\underline{\lambda}^*)\psi(\underline{x})$ for all specializations $\underline{\lambda} \to \underline{\lambda}^*$ in K^ℓ except possibly in a proper Zariski closed subset. It follows then from the Bertini-Krull theorem and the irreducibility of $\phi + \lambda\psi$ in $\overline{K(\lambda)}[\underline{x}]$ that $\mu(\underline{\lambda})$ has only finitely many specializations in K and so necessarily $\mu(\underline{\lambda}) = \mu \in K$. Then set $a(\underline{x}) = \phi(\underline{x}) + \mu\psi(\underline{x})$. In the case that $\alpha_k = 0$, set $a(\underline{x}) = \psi(\underline{x})$. In all cases, $a(\underline{x}) \in K[\underline{x}] \setminus K$ and $F(\underline{x},\underline{\lambda}) = a(\underline{x})G_{\underline{\lambda}}(\underline{x})$ for some $G_{\underline{\lambda}}(\underline{x}) \in \overline{K(\underline{\lambda})}[\underline{x}]$. We now show that this leads to a contradiction. Namely for each $i = 1,\dots,\ell$

$$\frac{\partial G_{\underline{\lambda}}}{\partial \lambda_i} = \frac{1}{a}\frac{\partial F}{\partial \lambda_i} = \frac{Q_i}{a}$$

lies both in $K(\underline{x})$ and in $\overline{K(\underline{\lambda})}[\underline{x}]$, and so is in $K[\underline{x}]$. Thus a divides Q_i in $K[\underline{x}]$, $i = 1,\dots,\ell$. But as a divides $P + \sum_{i=1}^{\ell} \lambda_i Q_i$, a divides P as well (both in $\overline{K(\underline{\lambda})}[\underline{x}]$): a contradiction as $\deg(a) > 0$ and $P, Q_1,\dots,Q_\ell$ are relatively prime.

It follows from claims (a) and (b) that if the two maximal homogeneous decompositions given in the statement of theorem 5.2 are reduced, then we have $(\phi_1,\psi_1) = L_{\underline{\lambda}}(\phi_2,\psi_2)$ for some $L_{\underline{\lambda}} \in \mathrm{GL}_2(\overline{K(\underline{\lambda})})$. Now for all $\underline{\lambda}^* \in K^\ell$ but in a proper Zariski closed subset we also have $(\phi_1,\psi_1) = L_{\underline{\lambda}^*}(\phi_2,\psi_2)$ with $L_{\underline{\lambda}^*} \in \mathrm{GL}_2(K)$.

It also follows from claims (a) and (b) that the set of linear factors $\alpha_k(\underline{\lambda})u + \beta_k(\underline{\lambda})v$ of the polynomial $H(u,v,\underline{\lambda})$ is uniquely determined (up to non zero constants) by the set of irreducible factors $\alpha_k(\underline{\lambda})\phi(\underline{x})+\beta_k(\underline{\lambda})\psi(\underline{x})$ of $F(\underline{x},\underline{\lambda})$. This yields the additional conclusion $c \cdot H_2(u,v,\underline{\lambda}) = H_1(u,v,\underline{\lambda}) \circ L_{\underline{\lambda}^*}(u,v)$ of theorem 5.2.

Finally if the two given maximal homogeneous decompositions of F are not reduced, consider the two associated reduced decompositions $F = H_1'(\phi_1,\psi_1,\underline{\lambda}) = H_2'(\phi_2,\psi_2,\underline{\lambda})$ (constructed prior to definition 5.1). The proof above still yields $(\phi_1,\psi_1) = L_{\underline{\lambda}^*}(\phi_2,\psi_2)$ for some $L_{\underline{\lambda}^*} \in \mathrm{GL}_2(K)$. □

5.1. **Further comments.** Retain the notation from the above proof.

5.1.1. As a consequence of the factors $\alpha_k(\underline{\lambda})\phi(\underline{x})+\beta_k(\underline{\lambda})\psi(\underline{x})$ not being in $K[\underline{x}]$ even up to constants in $\overline{K(\underline{\lambda})}$ we have $\alpha_k(\underline{\lambda})\beta_k(\underline{\lambda}) \neq 0$ and $\deg_{\underline{x}}(\alpha_k\phi + \beta_k\psi) = \max(\deg(\phi), \deg(\psi))$, $k = 1, \ldots, d$.

5.1.2. From the Bertini-Noether theorem [FrJa, proposition 8.8], for all $\underline{\lambda}^* \in K^\ell$ but in a proper Zariski closed subset $\mathcal{Z}$, the polynomials $\alpha_k(\underline{\lambda}^*)\phi(\underline{x}) + \beta_k(\underline{\lambda}^*)\psi(\underline{x})$, obtained by specializing $\underline{\lambda}$ to $\underline{\lambda}^*$ in the irreducible factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ of $F(\underline{x}, \underline{\lambda})$, are the irreducible factors of $F(\underline{x}, \underline{\lambda}^*)$ in $K[\underline{x}]$.

5.1.3. The vector space $\overline{K(\underline{\lambda})}\phi + \overline{K(\underline{\lambda})}\psi$, which is uniquely determined by $F(\underline{x}, \underline{\lambda})$, is the $\overline{K(\underline{\lambda})}$-vector space generated by all irreducible divisors of $F(\underline{x}, \underline{\lambda})$ in $\overline{K(\underline{\lambda})}[\underline{x}]$. As to the K-vector space $K\phi + K\psi$, it is the vector space generated by all irreducible divisors in $K[\underline{x}]$ of the polynomials $F(\underline{x}, \underline{\lambda}^*)$ with $\underline{\lambda}^* \notin \mathcal{Z}$ (where $\mathcal{Z}$ is defined just above).

5.1.4. Consider the general problem, given a polynomial P, of finding all the sets $\{Q_1, \ldots, Q_\ell\}$ ($\ell \geqslant 1$) of polynomials (not necessarily monomials) such that $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. We explain here how to reduce to the case $\ell = 1$.

If $\{Q_1, \ldots, Q_\ell\}$ is a solution to the general problem, then, for some integer $d \geqslant 2$, the polynomials $P, Q_1, \ldots, Q_\ell$ all are in the d-th symmetric power $(K\phi + K\psi)^d$ of some vector space $K\phi + K\psi \subset K[\underline{x}]$ [12] which from theorem 5.2 is uniquely determined by $P, Q_1, \ldots, Q_\ell$. Now there exists $Q \in (K\phi + K\psi)^d$ that is relatively prime to P. Clearly $P + \lambda Q$ is reducible in $\overline{K(\lambda)}[\underline{x}]$, that is, the singleton $\{Q\}$ is a solution to the problem with $\ell = 1$. The vector space $K\phi + K\psi$ is also uniquely determined by P and Q. Thus finding all solutions Q to the problem with $\ell = 1$ provides all possible solutions $\{Q_1, \ldots, Q_\ell\}$ to the general problem: these sets are all possible finite subsets of the sets $(K\phi + K\psi)^d$ attached to the solutions Q.

We note this other related consequence of theorem 5.2.

Corollary 5.5. *Suppose given two maximal homogeneous decompositions* $P(\underline{x}) + \lambda_1 Q_1(\underline{x}) + \cdots + \lambda_\ell Q_\ell(\underline{x}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ *and* $P(\underline{x}) + \lambda'_1 Q'_1(\underline{x}) + \cdots + \lambda'_{\ell'} Q'_{\ell'}(\underline{x}) = H'(\phi'(\underline{x}), \psi'(\underline{x}), \underline{\lambda}')$ *(with* $\ell, \ell' \geqslant 1$*). Assume further that* $Q_1 = Q'_1$ *and that* P *and* Q_1 *are relatively prime. Then we have* $(\phi', \psi') = L(\phi, \psi)$ *for some* $L \in \mathrm{GL}_2(K)$.

[12] This is another way of saying that each of these polynomials can be written $h(\phi, \psi)$ with $h \in K[u, v]$ homogeneous of degree d.

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