IRREDUCIBILITY OF HYPERSURFACES

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Abstract. Given a polynomial P in several variables over an algebraically closed field, we show that except in some special cases that we fully describe, if one coefficient is allowed to vary, then the polynomial is irreducible for all but at most $\deg(P)^2 - 1$ values of the coefficient. We more generally handle the situation where several specified coefficients vary.

1. Introduction

Classically polynomials in $n \geqslant 2$ variables are generically absolutely irreducible: if the coefficients, in some algebraically closed ground field K, are moved a little bit but stay away from some proper Zariski closed subset, then the resulting polynomial is irreducible over K. This is no longer true if only one specified coefficient is allowed to vary. For example however one moves a non-zero coefficient of some homogeneous polynomial $P(x, y) \in K[x, y]$ of degree $d \geqslant 2$, it remains reducible over K. Yet it seems that this case is exceptional and that most polynomials are irreducible up to moving any fixed coefficient away from finitely many values. This paper is aimed at making this more precise.

1.1. **The problem.** The problem can be posed in general as follows: given an algebraically closed field K (of any characteristic) and a polynomial $P \in K[\underline{x}]$ (with $\underline{x} = (x_1, \ldots, x_n)$), describe the "exceptional" *reducibility monomial sites* of P, that is those sets $\{Q_1, \ldots, Q_\ell\}$ of monomials in $K[\underline{x}]$ for which $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is *generically reducible*, *i.e.* reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$[1], where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ is a ℓ-tuple of independent indeterminates. When this is not the case, it follows from the Bertini-Noether theorem that the polynomial with shifted coefficients $P + \lambda_1^* Q_1 + \cdots + \lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$ for all $\underline{\lambda}^* = (\lambda_1^*, \ldots, \lambda_\ell^*)$ in a non-empty Zariski open subset of K^ℓ (and the converse is true).

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[1]Given a field k, we denote by $\overline{k}$ an algebraic closure of k.

The situation $\ell = 1$ has been extensively studied in the literature, notably for $Q_1 = 1$, that is when it is the constant term that is moved: see works of Ruppert [Ru], Stein [St], Ploski [Pl], Cygan [Cy], Lorenzini [Lo], Vistoli [Vi], Najib [Na], Bodin [Bo] et al. The central result in this case, which is known as Stein's theorem, is that $P + \lambda$ is generically irreducible if and only if $P(\underline{x})$ is not a composed polynomial[2] (some say "indecomposable"); furthermore, the so-called spectrum of P consisting of all $\lambda^* \in K$ such that $P + \lambda^*$ is reducible in $K[\underline{x}]$, which from Bertini-Noether is finite in this case, is of cardinality $< \deg(P)$. This was first established by Stein in two variables and in characteristic 0, then extended to all characteristics by Lorenzini and finally generalized to n variables by Najib. The result also extends to arbitrary monomials Q_1, and in fact to arbitrary polynomials [Lo] [Bo]; the indecomposability assumption should be replaced by the condition that P/Q_1 is not a composed rational function, and the bound $\deg(P)$ by $\deg(P)^2$.

1.2. **Our results.** We fully describe the reducibility monomial sites of polynomials in the general situation $\ell \geqslant 1$ (theorem 3.3). We obtain simple criteria for generic irreducibility, more practical than the previous indecomposability type conditions. These results can be combined with some ℓ-dimensional Stein-like description of the irreducibility set (proposition 4.1). Our contribution can be illustrated by the following three consequences.

Recall K is an algebraically closed field of any characteristic. Below by Newton representation of a polynomial in n variables we merely mean the subset of all points $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that the monomial $x_1^{a_1} \cdots x_n^{a_n}$ appears in the polynomial with a non-zero coefficient.

Theorem 1.1. *Let $P(\underline{x}) \in K[\underline{x}]$ be a non constant polynomial and $Q(\underline{x})$ be a monomial of degree $\leqslant \deg(P)$ and relatively prime to P. Assume that the monomials of P together with Q do not lie on a line in their Newton representation[3] and that Q is not a pure power[4] in $K[\underline{x}]$. Then $P + \lambda Q$ is generically irreducible and the set of all $\lambda^* \in K$ such that $P + \lambda^* Q$ is reducible in $K[\underline{x}]$ is finite of cardinality $< \deg(P)^2$.*

In particular a polynomial can always be made irreducible by changing only one of its coefficients provided it is not divisible by a non-constant monomial.

[2]that is, is not of the form $r(S(\underline{x}))$ with $S \in K[\underline{x}]$ and $r \in K[t]$ with $\deg(r) \geqslant 2$.

[3]The result also holds if P is a monomial (in which case P and Q are lined up in the Newton representation).

[4]We say a polynomial $R \in K[\underline{x}]$ is a *pure power* if there exist $S \in K[\underline{x}]$ and $e > 1$ such that $R = S^e$. The monomial $Q(\underline{x}) = x_1^{e_1} \cdots x_n^{e_n}$ is not a pure power if and only if $e_1, \dots, e_n$ are relatively prime.

The assumption on the monomials of P and Q is here to avoid what we call the exceptional homogeneous case, that is, that P be of the form $h(m_1, m_2)$ with $h \in K[u, v]$ homogeneous and m_1, m_2 two monomials of degree $< \deg(P)$, in which case for any monomial $Q = m_1^k m_2^{d-k}$ $(0 \leqslant k \leqslant d = \deg(h))$, $P + \lambda Q$ is generically reducible.

Pure power monomials Q, *e.g.* $Q = 1$, should also be excluded in theorem 1.1, but can nevertherless be dealt with under a slightly more general condition.

Theorem 1.2. *Let $P(\underline{x}) \in K[\underline{x}]$ be a non constant polynomial and $Q(\underline{x})$ be a monomial of degree $\leqslant \deg(P)$ and relatively prime to P. Assume P is not of the form $h(m, \psi)$ with $h \in K[u, v]$ an homogeneous polynomial, m a monomial dividing Q and $\psi \in K[\underline{x}]$ such that $\deg(P) > \max(\deg(m), \deg(\psi))$. Then $P + \lambda Q$ is generically irreducible and the set of all $\lambda^* \in K$ such that $P + \lambda^* Q$ is reducible in $K[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$.*

If P is of the excluded form then, for $Q = m^{\deg(h)}$, the polynomial $P + \lambda Q$ is generically reducible.

In the special case $Q = 1$, the assumption on P is that it is not of the form $h(1, \psi)$ with $h \in K[u, v]$ homogeneous, $\deg_v(h) \geqslant 2$ and $\psi \in K[\underline{x}]$: this corresponds to the classical hypothesis that P is not a composed polynomial. Thus theorem 1.2 is a generalization of Stein's theorem (except for the bound which can be taken to be $\deg(P)$ in this special case).

As another typical consequence of our approach, we obtain that for $\ell \geqslant 2$, reducibility monomials are even more rare.

Theorem 1.3. *Let $P \in K[\underline{x}]$ be a non constant polynomial and, for $\ell \geqslant 2$, $Q_1, \ldots, Q_\ell$ be ℓ monomials of degree $\leqslant \deg(P)$ and such that $P, Q_1, \ldots, Q_\ell$ are relatively prime. Assume the monomials of P together with $Q_1, \ldots, Q_\ell$ do not lie on a line in their Newton representation. If $\mathrm{char}(K) = p > 0$ assume further that at least one of $P, Q_1, \ldots, Q_\ell$ is not a p-th power. Then $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is generically irreducible and so $P + \lambda_1^* Q_1 + \cdots + \lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$ for all $(\lambda_1^*, \ldots, \lambda_\ell^*)$ in a non-empty Zariski open subset of K^ℓ.*[5]

For example $P(x_1, \ldots, x_n) + \lambda_1 x_1 + \cdots + \lambda_n x_n$ $(n \geqslant 2)$ is generically irreducible. See corollary 4.3 for further related results.

1.3. **Organization of the paper.** A starting ingredient of our method is the Bertini-Krull theorem, which gives an iff condition for some polynomial $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ to be generically irreducible. The

[5]Prop. 4.1 gives a more explicit Stein-like description of the irreducibility set.

Bertini-Krull theorem is recalled in the preliminary section 2 which also introduces some basic definitions used in the rest of the paper. We also seize the opportunity to prove a useful uniqueness result (theorem 2.7) in the Bertini-Krull theorem, which to our knowledge, was only known in the context of Stein's theorem.

Section 3 is the core of the paper. We investigate the Bertini-Krull conclusion in the specific context of our problem to finally obtain a general description of the reducibility monomial sites of a given polynomial (theorem 3.3). Giving an exact description requires controlling the possible overlaps of the special cases where reducibility monomial sites can exist. This comes down to proving (as in lemma 3.7) some uniqueness statements for "homogeneous decompositions" of polynomials related to those studied in section 2.

Section 4 is devoted to specializing the variables $\lambda_1, \dots, \lambda_\ell$. For $\ell = 1$, we use the generalization of Stein's theorem due to Lorenzini [Lo] and Bodin [Bo] to give an upper bound for the cardinality of the set of exceptional values λ^* making $P + \lambda^* Q$ reducible in $K[\underline{x}]$. A version of this estimate can be derived inductively for the situation $\ell \geqslant 1$, for which the classical Bertini-Noether theorem can also be used. We then complete the proof of the results from the introduction and give some further corollaries.

1.4. **Main Data and Notation.** The following is given and will be retained throughout the paper:

- an algebraically closed field K of characteristic 0 or $p > 0$,
- an integer $\ell \geqslant 0$ and an ℓ-tuple $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ of independent variables (algebraically independent over K); for $\ell = 0$, the convention is that no variable is given,
- an integer $n \geqslant 2$ and an n-tuple $\underline{x} = (x_1, \dots, x_n)$ of new independent variables (algebraically independent over $\overline{K(\underline{\lambda})}$),
- $\ell+1$ distinct (up to multiplicative constants) non-zero polynomials $P, Q_1, \dots, Q_\ell \in K[\underline{x}]$ with $\max(\deg(P), \dots, \deg(Q_\ell)) > 0$ and assumed further to be relatively prime if $\ell \geqslant 1$,
- $F(\underline{x}, \underline{\lambda}) = P(\underline{x}) + \lambda_1 Q_1(\underline{x}) + \cdots + \lambda_\ell Q_\ell(\underline{x})$, which is an irreducible polynomial in $K[\underline{x}, \underline{\lambda}]$ if $\ell \geqslant 1$. (For $\ell \geqslant 1$, $F(\underline{x}, \underline{\lambda})$ can be alternatively defined as a linear form in $(\lambda_0, \dots, \lambda_\ell)$ (with $\lambda_0 = 1$) with distinct non-zero and relatively prime coefficients in $K[\underline{x}]$).

2. Around the Bertini-Krull theorem

2.1. **Bertini-Krull theorem and homogeneous decompositions.** We start by recalling the Bertini-Krull theorem. We refer to [Sc, theorem 37] where equivalence between conditions (1) and (4) below is proved; equivalence between conditions (1), (2) and (3) is a special case of the standard Bertini-Noether theorem [FrJa, proposition 8.8].

Theorem 2.1 (Bertini, Krull)**.** *In addition to §1.4, assume $\ell \geqslant 1$. Then the following conditions are equivalent:*

(1) *$F(\underline{x}, \underline{\lambda}^*)$ is reducible in $K[\underline{x}]$ for all $\underline{\lambda}^* \in K^\ell$ such that* $\deg(F(\underline{x}, \underline{\lambda}^*)) = \deg_{\underline{x}}(F)$.

(2) *The set of $\underline{\lambda}^* \in K^\ell$ such that $F(\underline{x}, \underline{\lambda}^*)$ is reducible in $K[\underline{x}]$ is Zariski-dense.*

(3) *$F(\underline{x}, \underline{\lambda})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$.*

(4) (a) *either* $\operatorname{char} K = p > 0$ *and $F(\underline{x}, \underline{\lambda}) \in K[\underline{x}^p, \underline{\lambda}]$, where $\underline{x}^p = (x_1^p, \dots, x_n^p)$,*

(b) *or there exist $\phi, \psi \in K[\underline{x}]$ with* $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$ *satisfying the following:*
(*) there is an integer $d > 1$ [6] *and $\ell + 1$ polynomials $h_i(u, v) \in K[u, v]$ homogeneous of degree d such that*

$$\begin{cases} P(\underline{x}) = h_0(\phi(\underline{x}), \psi(\underline{x})) = \sum_{k=0}^{d} a_{0k}\phi(\underline{x})^k \psi(\underline{x})^{d-k} \\ Q_1(\underline{x}) = h_1(\phi(\underline{x}), \psi(\underline{x})) = \dots \\ \qquad\qquad\qquad\qquad \vdots \\ Q_\ell(\underline{x}) = h_\ell(\phi(\underline{x}), \psi(\underline{x})) = \sum_{k=0}^{d} a_{\ell k}\phi(\underline{x})^k \psi(\underline{x})^{d-k} \end{cases}$$

which, setting $H(u, v, \underline{\lambda}) = h_0(u, v) + \sum_{i=1}^{\ell} \lambda_i h_i(u, v)$, equivalently rewrites

$$F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda}).$$

Remark 2.2. (1) In (4a), it follows from $F(\underline{x}, \underline{\lambda}) \in K[\underline{x}^p, \underline{\lambda}]$ that $P, Q_1, \dots, Q_\ell$ are in $K[x_1^p, \dots, x_n^p]$; as K is algebraically closed they are also p-th powers in $K[\underline{x}]$.

(2) It follows from the assumption "$P, Q_1, \dots, Q_\ell$ relatively prime" that the same is true for ϕ and ψ in (4b).

The end of this section is devoted to the study of the decomposition $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ in (4b) (*) and particularly to the uniqueness of such a decomposition.

Definition 2.3. Given two polynomials $\phi, \psi \in K[\underline{x}]$ relatively prime and such that $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$,

(1) the polynomial F is said to be (ϕ, ψ)-*homogeneously composed (in degree d)* if there exists $H(u, v, \underline{\lambda}) \in \overline{K(\underline{\lambda})}[u, v]$ homogeneous (of degree d) in (u, v) such that $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$.

[6] This condition is actually a consequence of $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$.

The identity $F(\underline{x},\underline{\lambda}) = H(\phi(\underline{x}),\psi(\underline{x}),\underline{\lambda})$ is then called a (ϕ,ψ)-*homogeneous decomposition of* F. This definition is motivated by condition (4b) (*) of Bertini-Krull theorem.

(2) A (ϕ,ψ)-homogeneous decomposition $F(\underline{x},\underline{\lambda}) = H(\phi(\underline{x}),\psi(\underline{x}),\underline{\lambda})$ is said to be *maximal* if $\phi+\lambda\psi$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ [7].

Remark 2.4. (1) We also include in this definition the case $\ell = 0$ for which only the polynomial P is given. In this situation, the classical notion of composed polynomial corresponds to the special case of the "(ϕ,ψ)-homogeneously composed" property for which ϕ or ψ is constant.

(2) For $\ell \geqslant 1$ we will show that the *maximality* condition is equivalent (except in some special case) to the maximality of the degree of the homogeneous polynomial H, whence the terminology. See theorem 2.7 and corollary 2.9.

(3) From the Bertini-Krull theorem, "$\phi+\lambda\psi$ irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$" is equivalent to "$\phi+\lambda^*\psi$ irreducible in $K[\underline{x}]$ for at least one $\lambda^*\in K$ with $\deg(\phi+\lambda^*\psi) = \max(\deg(\phi),\deg(\psi))$" and also to "$\phi+\lambda^*\psi$ irreducible in $K[\underline{x}]$ for all but finitely many $\lambda^*\in K$".

The polynomial $F(x,y,\lambda) = x^4-\lambda y^4$ admits the (x^2,y^2)-homogeneous decomposition $F(x,y,\lambda) = H_1(x^2,y^2,\lambda)$ with $H_1(u,v,\lambda) = u^2-\lambda v^2$. It is not maximal as $x^2-\lambda y^2 = (x-\sqrt{\lambda}y)(x+\sqrt{\lambda}y)$. This decomposition however can be refined to a (x,y)-homogeneous decomposition, which is maximal: namely we have $F(x,y,\lambda) = H_2(x,y,\lambda)$ with $H_2(u,v,\lambda) = u^4-\lambda v^4$. This refinement is in fact always possible.

Proposition 2.5. *Assume $F(\underline{x},\underline{\lambda})$ is (ϕ_0,ψ_0)-homogeneously composed in degree d_0. Then there exists a maximal (ϕ,ψ)-homogeneous decomposition of F of degree $d\geqslant d_0$ and which is of degree $d>d_0$ if the initial decomposition is not maximal.*

Proof. Let $F(\underline{x},\underline{\lambda}) = H_0(\phi_0(\underline{x}),\psi_0(\underline{x}),\underline{\lambda})$ be a (ϕ_0,ψ_0)-homogeneous decomposition in degree d_0. If $\phi_0+\lambda\psi_0$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ then we are done. Otherwise apply the Bertini-Krull theorem to the polynomial $\phi_0+\lambda\psi_0$ (note that it is irreducible in $K[\lambda][\underline{x}]$ as ϕ_0 and ψ_0 are relatively prime) to conclude that there exist $\phi_1,\psi_1\in K[\underline{x}]$ relatively prime and with $\max(\deg(\phi_0),\deg(\psi_0)) > \max(\deg(\phi_1),\deg(\psi_1))$ such that $\phi_0+\lambda\psi_0$ is (ϕ_1,ψ_1)-homogeneously composed in degree $d_1\geqslant 2$. Note that this conclusion also covers the extra possibility (4a) of theorem 2.1 in characteristic $p>0$, which is here that $\phi_0+\lambda\psi_0$ writes $\phi_1^p+\lambda\psi_1^p$ for some $\phi_1,\psi_1\in K[\underline{x}]$. Straightforward calculations on homogeneous

[7] where λ is a new single variable (to be distinguished from the tuple $\underline{\lambda}$).

polynomials prove that F is then (ϕ_1, ψ_1)-homogeneously composed in degree $d_0 d_1 > d_0$. We can iterate this process, which must stop because at each step the degree increases but remains $\leqslant \deg_{\underline{x}}(F)$. The last step yields a final homogeneous decomposition of F which is maximal. $\square$

2.2. **Uniqueness of Bertini-Krull homogeneous decompositions.** Theorem 2.7, which can be viewed as a uniqueness result for the Bertini-Krull theorem, is the main result of this section. In this subsection, we assume $\ell \geqslant 1$.

We need a preliminary adjustment of definition 2.3. Given a (ϕ, ψ)-homogeneous decomposition $F(\underline{x}, \underline{\lambda}) = H(\phi, \psi, \underline{\lambda})$, assume there exists $(\alpha, \beta) \neq (0, 0)$ in K^2 such that $\alpha\phi + \beta\psi$ is constant in $\underline{x}$ (that is, is in K). Then multiplying $H(u, v, \underline{\lambda})$ by any power $(\alpha u + \beta v)^e$ yields another decomposition $F(\underline{x}, \underline{\lambda}) = \tilde{H}(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ as above with $\tilde{H}$ homogeneous (in u, v) of degree $\tilde{d} = d + e$. Conversely if $H(u, v, \underline{\lambda})$ has linear factors $\alpha u + \beta v$ (in $\overline{K(\underline{\lambda})}[u, v]$) with $\alpha\phi + \beta\psi$ constant in $\underline{x}$, then they are all equal, up to some constant in $\overline{K(\underline{\lambda})}$, to a same linear form $\alpha_0 u + \beta_0 v \in K[u, v]$ and the homogeneous polynomial $H'(u, v, \underline{\lambda})$ obtained from $H(u, v, \lambda)$ by dividing by all possible such factors $\alpha u+\beta v$ still induces a decomposition $F(\underline{x}, \underline{\lambda}) = H'(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ as above with H' homogeneous of degree $d' \leqslant d$. Note we still have $d' \geqslant 2$ as $d' \leqslant 1$ contradicts $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$.

Definition 2.6. Given two polynomials $\phi, \psi \in K[\underline{x}]$ relatively prime with $\deg_{\underline{x}}(F) > \max(\deg(\phi), \deg(\psi))$, a (ϕ, ψ)-homogeneous decomposition $F = H(\phi, \psi, \underline{\lambda})$ is said to be *reduced* if the polynomial H has no linear factor $\alpha u + \beta v \in K[u, v]$ such that $\alpha\phi + \beta\psi$ is constant in $\underline{x}$.

From above a reduced (ϕ, ψ)-homogeneous decomposition of F is easily obtained from any (ϕ, ψ)-homogeneous decomposition of F.

Also note that if there exists $(\alpha, \beta) \neq (0, 0)$ in K^2 such that $\alpha\phi+\beta\psi$ is constant, then up to applying some linear transformation $L \in \mathrm{GL}_2(K)$ to (ϕ, ψ), one may assume $\phi = 1$ and so this can only happen if F is a composed polynomial (over $\overline{K(\underline{\lambda})}$). Thus only in this case does definition 2.6 add something to definition 2.3.

Theorem 2.7. *Assume $\ell \geqslant 1$. If $F(\underline{x}, \underline{\lambda}) = P(\underline{x}) + \lambda_1 Q_1(\underline{x}) + \cdots + \lambda_\ell Q_\ell(\underline{x})$ admits two maximal homogeneous decompositions:*

$$F(\underline{x}, \underline{\lambda}) = H_1(\phi_1(\underline{x}), \psi_1(\underline{x}), \underline{\lambda}) = H_2(\phi_2(\underline{x}), \psi_2(\underline{x}), \underline{\lambda})$$

then there exists $L \in \mathrm{GL}_2(K)$ such that $(\phi_1, \psi_1) = L(\phi_2, \psi_2)$. Furthermore if the two decompositions are reduced then we have $c \cdot H_2(u, v, \underline{\lambda}) = H_1(u, v, \underline{\lambda}) \circ L(u, v)$ for some constant $c \in K$.

Example 2.8. Theorem 2.7 does not extend to the case $\ell = 0$. Here is a counter-example. Let $P(x,y) = y(x+y)(y^2+xy-2x)$. We have the two maximal homogeneous decompositions:
- $P = h_1(\phi_1, \psi_1)$ with $h_1(u,v) = v^2 - u^2$, $\phi_1 = x$, $\psi_1 = (y-1)(x+y)+y$,
- $P = h_2(\phi_2, \psi_2)$ with $h_2(u,v) = uv$, $\phi_2 = y$, $\psi_2 = (x+y)(y^2+xy-2x)$.
These two decompositions are distinct even up to elements of $\mathrm{GL}_2(K)$.

Corollary 2.9. *All reduced maximal homogeneous decompositions of F are of the same degree, say δ. Furthermore if F is not a composed polynomial over $\overline{K(\underline{\lambda})}$, any homogeneous decomposition of F is of degree $\leqslant \delta$ and equality holds if and only if it is maximal.*

Proof of theorem 2.7. Consider a reduced maximal homogeneous decomposition $F(\underline{x}, \underline{\lambda}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$. Write the homogeneous polynomial $H(u, v, \underline{\lambda})$ (in u, v) as a product $\prod_{i=1}^{d}(\alpha_i(\underline{\lambda})u + \beta_i(\underline{\lambda})v)$ of linear forms in u, v with coefficients in $\overline{K(\underline{\lambda})}$. Thus we have

$$P(\underline{x}) + \sum_{i=1}^{\ell} \lambda_i Q_i(\underline{x}) = \prod_{k=1}^{d}(\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})).$$

The result will be easily deduced from these two claims and the unique factorization property in the domain $\overline{K(\underline{\lambda})}[\underline{x}]$.

(a) There are at least two factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ that are non constant in $\underline{x}$ and non proportional (by some constant in $\overline{K(\underline{\lambda})}$).

(b) All factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ ($k = 1, \ldots, d$) are irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ and are not in $K[\underline{x}]$ (even up to constants in $\overline{K(\underline{\lambda})}$).

Proof of claim (a). First note that due to definition 2.6, no factor $\alpha_k\phi + \beta_k\psi$ is in $\overline{K(\underline{\lambda})}$. Assume (a) does not hold. Then $F(\underline{x}, \underline{\lambda})$ is of the form αM^d with $\alpha \in \overline{K(\underline{\lambda})}$ and $M = \alpha_1\phi + \beta_1\psi$. Taking the derivative with respect to λ_i shows that M^{d-1} divides Q_i in $\overline{K(\underline{\lambda})}[\underline{x}]$, $i = 1, \ldots, \ell$. But as M^d divides $F(\underline{x}, \underline{\lambda})$, we obtain that M^{d-1} divides P as well. A contradiction as $\deg(M) > 0$ and $P, Q_1, \ldots, Q_\ell$ are assumed to be relatively prime.

Proof of claim (b). Assume that for some $k \in \{1, \ldots, d\}$, $\alpha_k\phi + \beta_k\psi$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. One may assume that $\deg(\psi) > 0$ and $\beta_k \neq 0$. If $\alpha_k \neq 0$, set $\mu(\underline{\lambda}) = \beta_k(\underline{\lambda})/\alpha_k(\underline{\lambda})$. The polynomial $\phi(\underline{x}) + \mu(\underline{\lambda})\psi(\underline{x})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ and consequently so are the polynomials $\phi(\underline{x}) + \mu(\underline{\lambda}^*)\psi(\underline{x})$ for all specializations $\underline{\lambda} \to \underline{\lambda}^*$ in K^ℓ except possibly in a proper Zariski closed subset. It follows then from the Bertini-Krull theorem and the irreducibility of $\phi + \lambda\psi$ in $\overline{K(\lambda)}[\underline{x}]$ that $\mu(\underline{\lambda})$ has only finitely many specializations in K and so necessarily $\mu(\underline{\lambda}) = \mu \in K$.

Then set $a(\underline{x}) = \phi(\underline{x}) + \mu\psi(\underline{x})$. In the case that $\alpha_k = 0$, set $a(\underline{x}) = \psi(\underline{x})$. In all cases, $a(\underline{x}) \in K[\underline{x}] \setminus K$ and $F(\underline{x}, \underline{\lambda}) = a(\underline{x})G_{\underline{\lambda}}(\underline{x})$ for some $G_{\underline{\lambda}}(\underline{x}) \in \overline{K(\underline{\lambda})}[\underline{x}]$. We now show that this leads to a contradiction. Namely for each $i = 1, \ldots, \ell$

$$\frac{\partial G_{\underline{\lambda}}}{\partial \lambda_i} = \frac{1}{a}\frac{\partial F}{\partial \lambda_i} = \frac{Q_i}{a}$$

lies both in $K(\underline{x})$ and in $\overline{K(\underline{\lambda})}[\underline{x}]$, and so is in $K[\underline{x}]$. Thus a divides Q_i in $K[\underline{x}]$, $i = 1, \ldots, \ell$. But as a divides $P + \sum_{i=1}^{\ell} \lambda_i Q_i$, a divides P as well (both in $\overline{K(\underline{\lambda})}[\underline{x}]$): a contradiction as $\deg(a) > 0$ and $P, Q_1, \ldots, Q_\ell$ are relatively prime.

It follows from claims (a) and (b) that if the two maximal homogeneous decompositions given in the statement of theorem 2.7 are reduced, then we have $(\phi_1, \psi_1) = L_{\underline{\lambda}}(\phi_2, \psi_2)$ for some $L_{\underline{\lambda}} \in \mathrm{GL}_2(\overline{K(\underline{\lambda})})$. Now for all $\underline{\lambda}^* \in K^\ell$ but in a proper Zariski closed subset we also have $(\phi_1, \psi_1) = L_{\underline{\lambda}^*}(\phi_2, \psi_2)$ with $L_{\underline{\lambda}^*} \in \mathrm{GL}_2(K)$.

It also follows from claims (a) and (b) that the set of linear factors $\alpha_k(\underline{\lambda})u + \beta_k(\underline{\lambda})v$ of the polynomial $H(u, v, \underline{\lambda})$ is uniquely determined (up to non zero constants) by the set of irreducible factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ of $F(\underline{x}, \underline{\lambda})$. This yields the additional conclusion $c \cdot H_2(u, v, \underline{\lambda}) = H_1(u, v, \underline{\lambda}) \circ L_{\underline{\lambda}^*}(u, v)$ of theorem 2.7.

Finally if the two given maximal homogeneous decompositions of F are not reduced, consider the two associated reduced decompositions $F = H_1'(\phi_1, \psi_1, \underline{\lambda}) = H_2'(\phi_2, \psi_2, \underline{\lambda})$ (constructed prior to definition 2.6). The proof above still yields $(\phi_1, \psi_1) = L_{\underline{\lambda}^*}(\phi_2, \psi_2)$ for some $L_{\underline{\lambda}^*} \in \mathrm{GL}_2(K)$. □

2.3. **Further comments.** Retain the notation from the above proof.

2.3.1. As a consequence of the factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ not being in $K[\underline{x}]$ even up to constants in $\overline{K(\underline{\lambda})}$ we have $\alpha_k(\underline{\lambda})\beta_k(\underline{\lambda}) \neq 0$ and $\deg_{\underline{x}}(\alpha_k\phi + \beta_k\psi) = \max(\deg(\phi), \deg(\psi))$, $k = 1, \ldots, d$.

2.3.2. From the Bertini-Noether theorem [FrJa, proposition 8.8], for all $\underline{\lambda}^* \in K^\ell$ but in a proper Zariski closed subset $\mathcal{Z}$, the polynomials $\alpha_k(\underline{\lambda}^*)\phi(\underline{x}) + \beta_k(\underline{\lambda}^*)\psi(\underline{x})$, obtained by specializing $\underline{\lambda}$ to $\underline{\lambda}^*$ in the irreducible factors $\alpha_k(\underline{\lambda})\phi(\underline{x}) + \beta_k(\underline{\lambda})\psi(\underline{x})$ of $F(\underline{x}, \underline{\lambda})$, are the irreducible factors of $F(\underline{x}, \underline{\lambda}^*)$ in $K[\underline{x}]$.

2.3.3. The vector space $\overline{K(\underline{\lambda})}\phi + \overline{K(\underline{\lambda})}\psi$, which is uniquely determined by $F(\underline{x}, \underline{\lambda})$, is the $\overline{K(\underline{\lambda})}$-vector space generated by all irreducible divisors of $F(\underline{x}, \underline{\lambda})$ in $\overline{K(\underline{\lambda})}[\underline{x}]$. As to the K-vector space $K\phi + K\psi$, it is the vector space generated by all irreducible divisors in $K[\underline{x}]$ of the polynomials $F(\underline{x}, \underline{\lambda}^*)$ with $\underline{\lambda}^* \notin \mathcal{Z}$ (where $\mathcal{Z}$ is defined just above).

2.3.4. Consider the problem, given a polynomial P as above, of finding all the sets $\{Q_1, \dots, Q_\ell\}$ of polynomials as above (with $\ell \geqslant 1$), such that $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. This problem will be studied in the next section in the special situation $Q_1, \dots, Q_\ell$ are monomials. We note here that the general problem can be reduced to the special case $\ell = 1$.

Indeed, if $\{Q_1, \dots, Q_\ell\}$ is a solution to this problem, then, for some integer $d \geqslant 2$, the polynomials $P, Q_1, \dots, Q_\ell$ all are in the d-th symmetric power $(K\phi + K\psi)^d$ of some vector space $K\phi + K\psi \subset K[\underline{x}]$ [8] which from theorem 2.7 is uniquely determined by $P, Q_1, \dots, Q_\ell$. Now there exists $Q \in (K\phi + K\psi)^d$ that is relatively prime to P. Clearly $P + \lambda Q$ is reducible in $\overline{K(\lambda)}[\underline{x}]$, that is, the singleton $\{Q\}$ is a solution to the problem with $\ell = 1$. The vector space $K\phi + K\psi$ is also uniquely determined by P and Q. Thus finding all solutions Q to the problem with $\ell = 1$ provides all possible solutions $\{Q_1, \dots, Q_\ell\}$ to the general problem: these sets are all possible finite subsets of the sets $(K\phi + K\psi)^d$ attached to the solutions Q.

For self-containedness of next section, we will not use this remark there. We just state this other related consequence of theorem 2.7.

Corollary 2.10. *Suppose given two maximal homogeneous decompositions $P(\underline{x}) + \lambda_1 Q_1(\underline{x}) + \cdots + \lambda_\ell Q_\ell(\underline{x}) = H(\phi(\underline{x}), \psi(\underline{x}), \underline{\lambda})$ and $P(\underline{x}) + \lambda'_1 Q'_1(\underline{x}) + \cdots + \lambda'_{\ell'} Q'_{\ell'}(\underline{x}) = H'(\phi'(\underline{x}), \psi'(\underline{x}), \underline{\lambda}')$ (with $\ell, \ell' \geqslant 1$). Assume further that $Q_1 = Q'_1$ and that P and Q_1 are relatively prime. Then we have $(\phi', \psi') = L(\phi, \psi)$ for some $L \in \mathrm{GL}_2(K)$.*

3. Reducibility monomial sites

We keep the notation of section 2 but assume in addition that $\ell \geqslant 1$ and that $Q_1, \dots, Q_\ell$ are monomials such that $\deg(Q_i) \leqslant \deg(P)$, $i = 1, \dots, \ell$. We set $Q_i = x_1^{e_{i1}} \cdots x_n^{e_{in}}$, $i = 1, \dots, \ell$.

Definition 3.1. The set $\{Q_1, \dots, Q_\ell\}$ is said to be a *reducibility monomial site* of P is $F(\underline{x}, \underline{\lambda}) = P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. If $\ell = 1$ we just say Q_1 is a *reducibility monomial.*

It is readily checked that any subset of a reducibility monomial site is a reducibility monomial site.

[8]This is another way of saying that each of these polynomials can be written $h(\phi, \psi)$ with $h \in K[u, v]$ homogeneous of degree d.

Definition 3.2. A polynomial $P \in K[\underline{x}]$ is said to be *homogeneous in two monomials* if P is (m_1, m_2)-homogeneously composed for some monomials m_1 and m_2 (which according to definition 2.3 should be relatively prime and such that $\deg(P) > \max(\deg(m_1), \deg(m_2))$).

This property can be easily detected thanks to the Newton representation of P (as already used in the introduction). Indeed, set $m_1 = x_1^{a_1} \cdots x_n^{a_n}$ and $m_2 = x_1^{b_1} \cdots x_n^{b_n}$. If P is homogeneous in m_1 and m_2, then P is a sum of monomials of the form:

$$m_1^k m_2^{d-k} = x_1^{db_1+k(a_1-b_1)} \cdots x_n^{db_n+k(a_n-b_n)} \quad (k \in \{0, \dots, d\})$$

The corresponding points $M_k = (db_1 + k(a_1 - b_1), \dots, db_n + k(a_n - b_n))$ $(k = 0, \dots, d)$ lie on a straight line in $\mathbb{Q}^n$.[9]

We will show below (theorem 3.3 (addendum 1)) that a (m_1, m_2)-homogeneous decomposition of P is maximal, that is $m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ if and only if m_1 and m_2 are not d-th powers in $K[\underline{x}]$ for some integer $d > 1$, or, equivalently, if $a_1, \dots, a_n, b_1, \dots, b_n$ are relatively prime.

3.1. Main theorem.

Our main result determines the reducibility monomial sites of a polynomial. We first state it in the general situation of a polynomial that is neither a monomial nor a pure power. The two remaining special cases are dealt with in two addenda. The proof is given in section 3.5.

Theorem 3.3 (general case)**.** *Assume $P(\underline{x})$ is not a monomial and is not a pure power in $K[\underline{x}]$.*

(1) *If P is homogeneous in two monomials, then given a maximal (m_1, m_2)-homogeneous decomposition $P = h(m_1, m_2)$ of degree δ with m_1 and m_2 monomials*[10]*, the reducibility monomial sites of P are all sets of monomials $m_1^k m_2^{\delta-k}$, $0 \leqslant k \leqslant \delta$, of degree $\leqslant \deg(P)$.*

(2) *If P is not homogeneous in two monomials then the only possible reducibility monomial sites are singletons ($\ell = 1$) of the form $\{m^d\}$ with m a monomial relatively prime to P and $d \geqslant 2$.*

[9] Note however that the monomials being lined up in the Newton representation is not sufficient for P to be homogeneous in two monomials: for example $P = xy + x^2y^4 + x^3y^6$ has that property but is not homogeneous in two monomials. It is of course easy to give a full test for some polynomial P to be homogeneous in terms of its Newton representation but writing out the exact condition is not very enlightening. See also remark 3.8.

[10] Such a decomposition exists (proposition 2.5) and is unique up to trivial transformations (lemma 3.7).

Furthermore the following should hold: $P = h(m,\psi)$ *with* $h \in K[u,v]$ *homogeneous of degree* d*,* $\psi \in K[\underline{x}]$ *non monomial and* $\deg(P) > \max(\deg(m), \deg(\psi))$[11].

Remark 3.4. (1) In the homogeneous case (1), the reducibility monomials $m_1^k m_2^{\delta-k}$ also are on the line formed by the monomials of P in its Newton representation.

(2) In case (2) we do not know whether there may be several reducibility monomials of the form m^d. This is related to the possibility that P can be written $P = h(m,\psi)$ as in the statement in several different ways, and so to the uniqueness of homogeneous decompositions of P. In section 2.2 where this problem is studied for the polynomial $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ with $\ell \geqslant 1$, we give a counter-example to uniqueness for $\ell = 0$ (example 2.8). However the two monomials m^d associated to the two homogeneous decompositions of P shown there are x^2 and y^2; the second one is not relatively prime to P and so is not a reducibility monomial according to our definitions.

(3) In case (2) where $P = h(m,\psi)$, by setting $g(t) = h(1,t)$ we obtain $P/m^d = g(\psi/m)$ is a *composite rational function* as considered in [Bo] (of special form though as g is here a polynomial).

3.2. **The monomial case.** Here we consider the case P is a monomial $\gamma\, x_1^{e_1} \cdots x_n^{e_n}$ (with $\gamma \in K, \gamma \neq 0$). The argument below can be viewed as an easy special case of the general method.

From §2, if $F(\underline{x},\underline{\lambda})$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$, then equivalently either $F(\underline{x},\underline{\lambda}) \in K[\underline{x}^p,\underline{\lambda}]$ (with $\mathrm{char}(K) = p > 0$) or $F(\underline{x},\underline{\lambda})$ is (ϕ,ψ)-homogeneously composed in degree d for some $(\phi,\psi) \in K[\underline{x}]$. In the latter case, factor the homogeneous polynomials involved in the decomposition as products of linear forms to obtain

$$\begin{cases} P(\underline{x}) = \prod_{k=1}^{\mu_0} (\alpha_{0k}\phi(\underline{x}) + \beta_{0k}\psi(\underline{x}))^{r_{0k}} \\ Q_i(\underline{x}) = \prod_{k=1}^{\mu_i} (\alpha_{ik}\phi(\underline{x}) + \beta_{ik}\psi(\underline{x}))^{r_{ik}} \quad (i = 1,\ldots,\ell) \end{cases}$$

where the (α_{ik},β_{ik}) are non-zero, pairwise non proportional and the integers r_{ik} are > 0 and satisfy $\sum_{k=1}^{\mu_i} r_{ik} = d$ $(i = 0,\ldots,\ell)$.

All the factors appearing in the right-hand side terms are necessarily monomials and at least two of them are non proportional (as $P, Q_1, \ldots, Q_\ell$ are relatively prime). Therefore up to changing (ϕ,ψ) to $L(\phi,\psi)$ for some $L \in \mathrm{GL}_2(K)$ one may assume that ϕ and ψ themselves are two monomials m_1 and m_2. Taking into account that $P, Q_1, \ldots, Q_\ell$

[11]By proposition 2.5 we may also impose that $\psi + \lambda m$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$.

are monomials and that they are relatively prime, we obtain the following characterization (the converse is clear).

Theorem 3.3 (addendum 1). *If P is a monomial the following are equivalent:*

(1) *The polynomial $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ (that is, $\{Q_1, \ldots, Q_\ell\}$ is a reducibility monomial site of P),*

(2) (a) *either* $\operatorname{char} K = p > 0$ *and $P, Q_1, \ldots, Q_\ell \in K[\underline{x}^p]$,*

(b) *or $P, Q_1, \ldots, Q_\ell$ are of the form $m_1^k m_2^{d-k}$ ($0 \leqslant k \leqslant d$) for some relatively prime monomials m_1 and m_2 and some integer $d > 1$, and they include m_1^d and m_2^d.*
Furthermore, for all (ϕ, ψ)-homogeneous decompositions of $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$, (ϕ, ψ) is a couple of monomials, up to some element $L \in \mathrm{GL}_2(K)$.

Remark 3.5. In general there may be several couples (m_1, m_2) such that P is of the form $m_1^k m_2^{d-k}$, and so several corresponding reducibility sites for P. For example $P = x^3 y^2$ is homogeneously composed for both couples of monomials (x^3, y^2) and $(x^3 y^2, 1)$ and both decompositions are maximal. In the non monomial case, this will not happen: up to trivial transformations the couple (m_1, m_2) is uniquely determined by P (see lemma 3.7).

3.3. **Pure power case.** In the case P is a pure power in $K[\underline{x}]$, the three following possibilities can occur:

(1) P is homogeneous in two monomials. In this case let $P = h(m_1, m_2)$ be a maximal homogeneous decomposition of degree δ in two monomials m_1 and m_2 and set $\mathcal{M}_1 = \{m_1^k m_2^{\delta-k} \mid 0 \leqslant k \leqslant \delta\}$. All subsets of $\mathcal{M}_1$ are reducible monomial sites.

(2) P admits a maximal (m, ψ)-homogeneous decomposition in degree d, with m a monomial and $\psi \in K[\underline{x}]$ non monomial. In this case, if $\deg(m^d) \leqslant \deg(P)$, then m^d is a reducibility monomial.

(3) $\operatorname{char}(K) = p > 0$ and $P \in K[\underline{x}^p]$. In this case set $\mathcal{M}_3 = \{m^p \mid m \text{ is a monomial and} \deg(m^p) \leqslant \deg(P)\}$. All subsets of $\mathcal{M}_3$ are reducible monomial sites.

Theorem 3.3 (addendum 2). *Assume P is a pure power but is not a monomial. Then the reducibility monomial sites of P are those described in possibilities (1), (2) and (3).*

The following observations make the pure power case rather special:

(a) possibility (2) is always satisfied: indeed by assumption we have $P = S^e$ for some $S \in K[\underline{x}]$ and some integer $e > 1$, which is a (m, S)-homogeneous decomposition of degree e for any monomial m relatively prime to S; the corresponding monomials m^e with $\deg(m^e) \leqslant \deg(P)$ are reducibility monomials. However there may be other kinds of decompositions $P = h(m, \psi)$. For example, take $P(x, y) = (2y^3 - x^4)^2 x^4$. Squares monomials of degree $\leqslant 12$ are reducibility monomials. Now for $m = y^3$, $\psi = y^3 - x^4$ and $h(u, v) = (u + v)^2(u - v)$, we also have $P = h(m, \psi)$ and so $m^3 = y^9$ is another reducibility monomial of P.

(b) possibilities (1), (2) and (3) can occur simultaneously. Take for example $P(x, y) = (x^2 - y^3)^3$. Then P is homogeneous in the two monomials x^2 and y^3; the corresponding set $\mathcal{M}_1$ is $\mathcal{M}_1 = \{x^6, x^4y^3, x^2y^6, y^9\}$. As P is a third power, each of the monomials 1, x^3, y^3, x^6, x^3y^3, y^6, x^9, x^6y^3, x^3y^6, y^9 is a reducibility monomial. Finally if $\mathrm{char}(K) = 3$, then every subset of $\mathcal{M}_3 = \{1, x^3, y^3, x^6, x^3y^3, y^6, x^9, x^6y^3, x^3y^6, y^9\}$ is a reducibility monomial site.

3.4. **Lemmas.** The two following lemmas will be used in the proof of theorem 3.3.

Lemma 3.6. *Given two monomials $m_1, m_2 \in K[\underline{x}]$ such that we have* $\max(\deg(m_1), \deg(m_2)) > 0$*, the following are equivalent:*

(i) *there exists $\lambda^* \in K$, $\lambda^* \neq 0$, such that $m_1 + \lambda^* m_2$ is irreducible in $K[\underline{x}]$,*
(ii) *for all $\lambda^* \in K$, $\lambda^* \neq 0$, $m_1 + \lambda^* m_2$ is irreducible in $K[\underline{x}]$,*
(iii) *$m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$.*

Proof. The equivalence (iii)⇔(i) is a special case of the Bertini-Krull theorem and (ii)⇒(i) is trivial. We are left with proving (i)⇒(ii). Assume there exist $\lambda_1^*, \lambda_2^* \in K$, both non zero and such that $m_1 + \lambda_1^* m_2$ is reducible and $m_1 + \lambda_2^* m_2$ is irreducible in $K[\underline{x}]$.

Set $m_1 = x_1^{a_1} \cdots x_n^{a_n}$ and $m_2 = x_1^{b_1} \cdots x_n^{b_n}$. One may assume that $\deg(m_2) > 0$ and so for example $b_1 > 0$. If $a_1 > 0$ then x_1 divides $m_1 + \lambda_2^* m_2$ and so $m_1 = m_2 = x_1$ (up to some non-zero multiplicative constants) in which case the result is obvious. Thus one may assume $a_1 = 0$. If $m_1(\underline{x}) + \lambda_1^* m_2(\underline{x}) = R(\underline{x}) \cdot S(\underline{x})$ is a non trivial factorization of $m_1 + \lambda_1^* m_2$ $(\deg(R), \deg(S) > 0)$, we have

$$m_1 + \lambda_2^* m_2 = R\left((\lambda_1^{*-1}\lambda_2^*)^{\frac{1}{b_1}} x_1, x_2, \ldots, x_n\right) \cdot S\left((\lambda_1^{*-1}\lambda_2^*)^{\frac{1}{b_1}} x_1, x_2, \ldots, x_n\right)$$

which contradicts the irreducibility of $m_1 + \lambda_2^* m_2$. □

Lemma 3.7. *Assume $P(\underline{x})$ is not a monomial and is given with a maximal (m_1, m_2)-homogeneous decomposition $P = h(m_1, m_2)$ of degree d with m_1 and m_2 monomials.*

(1) *If $P = h'(m_1', m_2')$ is another maximal homogeneous decomposition of degree d' of P in monomials m_1' and m_2', then either*

($m_1 = am'_1$ and $m_2 = bm'_2$) or ($m_1 = am'_2$ and $m_2 = bm'_1$), for some non-zero constants $a, b \in K$, and $d = d'$.

(2) *There is no maximal homogeneous (m, ψ)-decomposition of $P = h'(m, \psi)$ with $\psi \in K[\underline{x}]$ non monomial and m a monomial relatively prime to P and not a monomial of ψ unless $P = \psi^{d''}$ with ψ homogeneous in m_1 and m_2 and $d'' \geqslant 2$.*

Proof. We can write

$$(**) \qquad P = h(m_1, m_2) = \prod_{k=1}^{\mu} (\alpha_k m_1 + \beta_k m_2)^{r_k}$$

where the (α_k, β_k) are non-zero and pairwise non-proportional and the integers r_k are > 0 and satisfy $\sum_{k=1}^{\mu} r_k = d$.

(1) As P is not a monomial there exists $k \in \{1, \ldots, \mu\}$ such that $\alpha_k \beta_k \neq 0$. Then by lemma 3.6 $\alpha_k m_1 + \beta_k m_2$ is irreducible in $K[\underline{x}]$.

Assume P has another maximal homogeneous decomposition in monomials m'_1 and m'_2

$$P = h'(m'_1, m'_2) = \prod_{k=1}^{\mu'} (\alpha'_k m'_1 + \beta'_k m'_2)^{r'_k}$$

where the (α'_k, β'_k) are non-zero, pairwise non proportional and the integers r'_k are > 0 and satisfy $\sum_{k=1}^{\mu'} r'_k > 1$. From the unique factorization property in the domain $K[\underline{x}]$, there exists $h \in \{1, \ldots, \mu'\}$ with $\alpha'_h \beta'_h \neq 0$ such that, up to a non-zero multiplicative constant, we have $\alpha_k m_1 + \beta_k m_2 = \alpha'_h m'_1 + \beta'_h m'_2$. As m_1, m_2, m'_1, m'_2 are monomials we obtain the desired conclusion.

Remark 3.8. In fact the monomials m_1 and m_2 of some maximal homogeneous decomposition of P can be easily recovered from the Newton representation of P. Indeed, using the notation from the beginning of section 3, for any two distinct points M_h and M_k, we have $\overrightarrow{M_k M_h} = (k-h)\overrightarrow{\Delta}$ where $\overrightarrow{\Delta} = (a_1 - b_1, \ldots, a_n - b_n)$. As $\min(a_j, b_j) = 0$, $j = 1, \ldots, \ell$, the non-zero exponents of m_1 (resp. of m_2) correspond to the positive components (resp. to the negative components) of $\overrightarrow{\Delta}$. As $a_1, \ldots, a_n, b_1, \ldots, b_n$ are relatively prime, these exponents correspond to the components of $\overrightarrow{M_k M_h}$ divided by their g.c.d.

(2) Suppose P has a maximal (m, ψ)-homogeneous decomposition (with m and ψ as in the statement)

$$P = h'(m, \psi) = \prod_{k=1}^{\mu'} (\alpha'_k \psi + \beta'_k m)^{r'_k}$$

where the (α'_k, β'_k) are non-zero and pairwise non-proportional and the integers r'_k are > 0 and satisfy $\sum_{k=1}^{\mu'} r'_k = d' > 1$.

Consider first the case there exists $h \in \{1, \ldots, \mu'\}$ with $\alpha'_h \beta'_h \neq 0$. Comparing with $(**)$ above we obtain that the polynomial $\alpha'_h \psi + \beta'_h m$ is a product of say ν irreducible factors $\alpha_k m_1 + \beta_k m_2$ with $\alpha_k \beta_k \neq 0$ (irreducible by lemma 3.6) and possibly some monomial ρ. As $\alpha'_h \psi + \beta'_h m$ has at least 3 monomials, the integer ν is $\geqslant 2$. Thus $\alpha'_h \psi + \beta'_h m$ can be written $\rho\, \kappa(m_1, m_2)$ with $\kappa \in K[u, v]$ homogeneous of degree $\nu \geqslant 2$. As m is not a monomial of ψ, conclude that, up to non zero constants in K, m is one of the monomials of $\kappa(m_1, m_2)$ multiplied by ρ and that ψ is the sum of the other monomials of $\kappa(m_1, m_2)$, also multiplied by ρ. Now as ψ and m are relatively prime, ρ is a non-zero constant in K. But then $m + \lambda\psi$ is (m_1, m_2)-homogeneously composed in degree ν, which contradicts the maximality of the (m, ψ)-decomposition.

Assume next that $\alpha'_h \beta'_h = 0$ for all $h = 1, \ldots, \mu'$. If no coefficient α'_h is zero, then $P = \psi^{d'}$ (up to some non-zero multiplicative constant). If some coefficient α'_h is zero, then m divides P and as P and m are assumed to be relatively prime, m is a non-zero constant in K. Conclude in both cases that $P = \psi^{d''}$ (up to some non-zero multiplicative constant) where d'' is the number of coefficients α'_h that are non-zero (counted with the multiplicities r'_k); we have $d'' \leqslant d'$ and $d'' \geqslant 2$ for otherwise we would have $\deg(P) \leqslant \max(\deg(\psi), \deg(m))$. Observe next that the exponents r_k are all divisible by d'': if $\alpha_k \beta_k \neq 0$, this is because $\alpha_k m_1 + \beta_k m_2$ is irreducible in $K[\underline{x}]$ and for the possible two factors that are powers of m_1 and m_2, because m_1 and m_2 are relatively prime. Conclude that ψ is as announced homogeneous in m_1 and m_2. □

3.5. **Proof of theorem 3.3.** Addendum 1 has already been proved (in section 3.2) so we may assume P is not a monomial.

3.5.1. *Preliminary discussion:* Let $\{Q_1, \ldots, Q_\ell\}$ $(\ell \geqslant 1)$ be a reducibility monomial site of P.

From remark 2.2 the case (4a) in the Bertini-Krull theorem can only occur if P is a pure power, and in this case the conclusion corresponds to possibility (3) of theorem 3.3 (addendum 2).

Suppose now it is part (4b) of the Bertini-Krull theorem that holds. That is, the polynomial $F(\underline{x}, \underline{\lambda}) = P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ has a (ϕ, ψ)-homogeneous decomposition in degree d for some $\phi, \psi \in K[\underline{x}]$, which in addition we may and will assume to be maximal (proposition 2.5).

Thus we have $P(\underline{x}) = h_0(\phi(\underline{x}), \psi(\underline{x}))$ and $Q_i(\underline{x}) = h_i(\phi(\underline{x}), \psi(\underline{x}))$ $(i = 1, \ldots, \ell)$ for some homogeneous polynomials $h_0, \ldots, h_\ell \in K[u, v]$ of degree d. Note that as $\deg(P) \geqslant \deg(Q_i)$, $i = 1, \ldots, \ell$, we have $\deg_{\underline{x}}(F) = \deg(P) > \max(\deg(\phi), \deg(\psi))$ and so $P = h_0(\phi, \psi)$ is still a (ϕ, ψ)-homogeneous decomposition of P. Write then $h_i(u, v) = \prod_{k=1}^{\mu_i} (\alpha_{ik} u + \beta_{ik} v)^{r_{ik}}$ with, for each $i = 0, \ldots, \ell$, the $(\alpha_{ik}, \beta_{ik})$ non-zero and pairwise non proportional and the integers $r_{ik} > 0$ and satisfying $\sum_{k=1}^{\mu_i} r_{ik} = d$. Unless $\ell = 1$ and Q_1 is constant, one may assume Q_1 is a non constant monomial and then all factors $\alpha_{1k}\phi(\underline{x}) + \beta_{1k}\psi(\underline{x})$ $(k = 1, \ldots, \mu_1)$ are monomials and at least one, say m, is non constant. If $\ell = 1$ and Q_1 is constant, then ϕ or ψ, say ϕ is constant. In all cases, up to changing (ϕ, ψ) to $L(\phi, \psi)$ for some $L \in \mathrm{GL}_2(K)$, one may assume that ϕ is a monomial m and that m is not a monomial of ψ. Observe then that if ψ has at least two monomials then $Q_i = h_i(m, \psi)$ can be a monomial only if $h_i(u, v) = u^d$ and so $\ell = 1$ and $Q_1 = m^d$.

We now distinguish two cases.

3.5.2. *1st case:* P is homogeneous in two monomials.

Let $P = h(m_1, m_2)$ be a maximal (m_1, m_2)-homogeneous decomposition in degree δ with m_1 and m_2 monomials. From above $P = h_0(m, \psi)$ is another maximal homogeneous decomposition.

If ψ itself is a monomial then from lemma 3.7 (1), we have $d = \delta$ and $(m, \psi) = (am_1, bm_2)$ or $(m, \psi) = (bm_2, am_1)$ for some non-zero constants $a, b \in K$. Conclude each Q_i is homogeneous in m_1 and m_2 in degree δ and as Q_i is a monomial, it should be of the form $m_1^k m_2^{\delta - k}$ for some $k \in \{0, \ldots, \delta\}$. Conversely, any set consisting of such monomials is clearly a reducibility monomial site of P.

Assume next that ψ is not a monomial. From the preliminary discussion $\ell = 1$ and $Q_1 = m^d$. In particular, P and m are relatively prime. It follows from lemma 3.7 (2) that $P = \psi^{d''}$ with ψ homogeneous in m_1 and m_2 and $d'' \geqslant 2$. In particular this can only occur if P is a pure power. Thus we are done with case (1) of theorem 3.3 (general) where P being a pure power is excluded. If P is a pure power, what we have obtained is contained in possibilities (1) and (2) from theorem 3.3 (addendum 2).

3.5.3. *2nd case: P is not homogeneous in two monomials.*

In this case ψ is not a monomial and the desired conclusions — that is, on one hand, case (2) of theorem 3.3 (general) and on the other hand that only possibility (2) can occur apart from possibilities (1) and (3) in theorem 3.3 (addendum 2) — are part of the preliminary discussion.

4. Specialization

In this section we explain how irreducibility properties of $F(\underline{x}, \underline{\lambda})$ can be preserved by specialization of the variables λ_i in K. This is the last stage towards the results stated in the introduction.

4.1. **Using Stein like results.**

Proposition 4.1. *Assume $F(\underline{x}, \underline{\lambda}) = P+\lambda_1 Q_1+\cdots+\lambda_\ell Q_\ell$ is irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$ (that is $\{Q_1, \ldots, Q_\ell\}$ is not a reducibility monomial site of P). Then for every $i = 1, \ldots, \ell$, the set of $\lambda_i^* \in K$ such that $P + \lambda_1 Q_1 + \cdots + \lambda_{i-1} Q_{i-1} + \lambda_i^* Q_i + \lambda_{i+1} Q_{i+1} + \cdots + \lambda_\ell Q_\ell$ is reducible in $\overline{K(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_\ell)}[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$.*

Consequently, for every $\lambda_1^* \in K$ but in a finite set of cardinality $< \deg(P)^2$, for every $\lambda_2^* \in K$ but in a finite set of cardinality $< \deg(P)^2$ (depending on λ_1^*),..., for every $\lambda_\ell^* \in K$ but in a finite set of cardinality $< \deg(P)^2$ (depending on $\lambda_1^*, \ldots, \lambda_{\ell-1}^*$), the polynomial $P+\lambda_1^* Q_1+\cdots+\lambda_\ell^* Q_\ell$ is irreducible in $K[\underline{x}]$.

Remark 4.2. The assumption "$P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$" holds if it holds for a smaller ℓ, in particular if P itself is irreducible in $K[\underline{x}]$. This follows immediately from the equivalence of (1) and (3) in the Bertini-Krull theorem.

Proof of proposition 4.1. With no loss of generality we may assume $i = 1$ in the first part. Set $G = P+\lambda_2 Q_2+\cdots+\lambda_\ell Q_\ell$ and $L = \overline{K(\lambda_2, \ldots, \lambda_\ell)}$. By hypothesis, $G+\lambda_1 Q_1$ is irreducible in $\overline{L(\lambda_1)}[\underline{x}]$. From the generalization of Stein's theorem to general pencils of hypersurfaces $P+\lambda Q$ (and not just the curves $P+\lambda$) given in [Bo] (relying on [Ru], [Lo] and [Na]), the set of $\lambda^* \in F$ such that $G + \lambda^* Q_1$ is reducible in $L[\underline{x}]$ is finite and of cardinality $< \deg(P)^2$. The second part is an easy induction. □

4.2. **Proof of the results from the introduction.**

4.2.1. *Proof of theorem 1.1.* Due to the assumptions on the monomials of P and Q, Q cannot be a reducibility monomial in the homogeneous case (1) from theorem 3.3 (general) nor in possibility (1) from theorem 3.3 (addendum 2). The monomial Q not being a pure power forbids condition (2) from theorem 3.3 (addendum 1) (with $\ell = 1$ and $Q_1 =$

Q) to happen and Q to be a reducibility monomial in case (2) from theorem 3.3 (general) and in possibilities (2) and (3) from theorem 3.3 (addendum 2). Therefore $P+\lambda Q$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$. Apply then proposition 4.1 to complete the proof of theorem 1.1.

4.2.2. *Proof of theorem 1.2.* Assume as in theorem 1.2 that P is not of the form $h(m,\psi)$ with $h \in K[u,v]$ homogeneous of degree $\geqslant 2$, $\psi \in K[\underline{x}]$ and m a monomial dividing Q. In particular P is not a pure power (for otherwise P is of this form with $h(u,v) = v^d$ for some $d > 1$ and $m = 1$). We show below that assuming Q is a reducibility monomial of P leads to a contradiction.

The homogeneous case (1) from theorem 3.3 (general) can be ruled out as follows. If this case occured, then by assumption neither m_1 nor m_2 could divide Q but this is not possible in view of the form of the reducibility monomial sites in this case.

The case P is a monomial can also be excluded: condition (2) from theorem 3.3 (addendum 1) (with $\ell = 1$ and $Q_1 = Q$) cannot hold since P is not a pure power.

The remaining possibility (2) from theorem 3.3 (general) cannot happen either since in this case P should be of the form $h(m,\psi)$ as above and $Q = m^d$ (and so m divides Q).

Conclude Q is not a reducibility monomial of P, that is, $P + \lambda Q$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$, and apply proposition 4.1 to complete the proof of theorem 1.2.

4.2.3. *Proof of theorem 1.3.* Here $\ell \geqslant 2$. The reducibility monomial sites of cardinality ℓ can only occur in the homogeneous cases from theorem 3.3 or in characteristic $p > 0$. But these possibilities are ruled out by the assumptions. Therefore $P + \lambda_1 Q_1 + \cdots + \lambda_\ell Q_\ell$ is irreducible in $\overline{K(\underline{\lambda})}[\underline{x}]$. Apply then the classical Bertini-Noether theorem [FrJa, proposition 8.8] or alternatively proposition 4.1 to conclude the proof.

4.3. **Further consequences.** We give below some variations around Stein's theorem which can be deduced from our results.

Corollary 4.3. *Let $P \in K[x_1,\ldots,x_n]$ be a polynomial in $n \geqslant 2$ variables and with coefficients in the algebraically closed field K.*

(1) *If P is not a composed polynomial then $P(x_1,\ldots,x_n) + \lambda^*$ is irreducible for all but at most* $\deg(P) - 1$ *values of $\lambda^* \in K$.*
(2) *If $P \notin K[x_1]$ and is not divisible by x_1, then $P(x_1,\ldots,x_n)+\lambda^* x_1$ is irreducible for all but at most* $\deg(P)^2 - 1$ *values of $\lambda^* \in K$.*

(3) *If* $P \in K[x_2, \ldots, x_n]$ *is not a pure power and* e *is an integer such that* $0 < e \leqslant \deg P$ *then* $P(x_2, \ldots, x_n) + \lambda^* x_1^e$ *is irreducible for all but at most* $\deg(P)^2 - 1$ *values of* $\lambda^* \in K$.

(4) *If* $n = 2$ *and* $P(x, y) \in K[x, y]$ *is homogeneous of degree* $d > 1$ *but is not a pure power and* $Q = x^i y^j$ *is a monomial of degree* $i + j < d$ *and relatively prime to* P, *then* $P(x, y) + \lambda^* x^i y^j$ *is irreducible for all but at most* $\deg(P)^2 - 1$ *values of* $\lambda^* \in K$.

Proof. (1) This is the special case $Q = 1$ of theorem 1.2 (see the comment after theorem 1.2). The bound for the number of exceptional values λ^* is obtained by using Stein's theorem [St] instead of the general bound from [Bo] as in proposition 4.1.

(2) Suppose that $P(x_1, \ldots, x_n) + \lambda x_1$ is reducible in $\overline{K(\lambda)}[\underline{x}]$. As x_1 is not a pure power, it follows from theorem 3.3 that $P = h(m_1, m_2)$ for some homogeneous polynomial $h \in K[u, v]$ of degree $d > 1$ and some monomials m_1 and m_2 and that $x_1 = m_1^k m_2^{d-k}$ for some $k \in \{0, \ldots, d\}$. Then we have necessarily $\{m_1, m_2\} = \{1, x_1\}$. But then $P = h(m_1, m_2)$ contradicts the assumption $P \notin K[x_1]$. Thus $P(x_1, \ldots, x_n) + \lambda x_1$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$ and the result follows from proposition 4.1.

(3) We first show that $P(x_2, \ldots, x_n) + \lambda x_1^e$ is irreducible in $\overline{K(\lambda)}[\underline{x}]$. From theorem 3.3 we need to exclude the two following situations.

(1) $P = h(m_1, m_2)$ for some homogeneous polynomial $h \in K[u, v]$ of degree $d > 1$ and some relatively prime monomials m_1 and m_2 and $x_1^e = m_1^k m_2^{d-k}$ with $0 \leqslant k \leqslant d$. If $0 < k < d$ then necessarily one of the two monomials, say m_1, is constant and m_2 is a pure power of x_1. But then $P = h(m_1, m_2)$ contradicts the assumption $\deg_{x_1}(P) = 0$. If $k = 0$, m_2 is a pure power of x_1 but then $P = h(m_1, m_2)$ is possible only if $P = m_1^d$ (for otherwise $\deg_{x_1}(P) > 0$), which is excluded as P is not a pure power. The case $k = d$ is similar.

(2) $P = h(m, \psi)$ for some homogeneous polynomial $h \in K[u, v]$ of degree $d > 1$ and $x_1^e = m^d$. Then m is a pure power of x_1 and as above $P = h(m, \psi)$ is possible only if $P = \psi^d$ (for otherwise $\deg_{x_1}(P) > 0$), which is excluded as P is a not pure power.

The result follows then from proposition 4.1.

(4) Irreducibility of $P(x, y) + \lambda x^i y^j$ in $\overline{K(\lambda)}[x, y]$ readily follows from theorem 3.3 (general & addendum 1): just note P is homogeneous in the two monomials $m_1 = x$ and $m_2 = y$, which are relatively prime, of degree $< \deg(P)$ and such that $m_1 + \lambda m_2$ is irreducible in $\overline{K(\lambda)}[x, y]$. Apply then proposition 4.1 to complete the proof. $\square$

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