THE BRAID GROUP OF A NECKLACE

PAOLO BELLINGERI AND ARNAUD BODIN

Abstract. We show several geometric and algebraic aspects of a *necklace*: a link composed with a core circle and a series of circles linked to this core. We first prove that the fundamental group of the configuration space of necklaces (that we will call braid group of a necklace) is isomorphic to the braid group over an annulus quotiented by the square of the center. We then define braid groups of necklaces and affine braid groups of type $\mathcal{A}$ in terms of automorphisms of free groups and characterize these automorphisms among all automorphisms of free groups. In the case of affine braid groups of type $\mathcal{A}$ such representation is faithful.

1. Introduction

The braid group B_n can be defined as the fundamental group of the configuration space of n distinct points on a disk. By extension, when we replace the disk by another surface Σ, we define the *braid group on n strands over Σ* as the fundamental group of the configuration space of n distinct points on Σ. A particular case is when Σ is the annulus: the braid group of the annulus, denoted by CB_n (for *circular braid group*) is the fundamental group of the configuration space $\mathcal{CB}_n$ of n distinct points over an annulus.

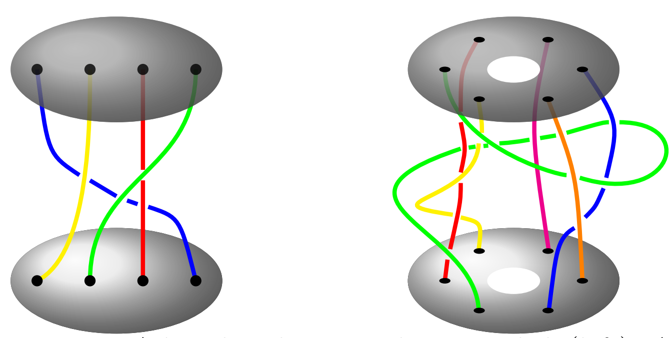

Figure 1. A braid with 4 strands over a disk (left). A braid with 6 strands over an annulus (right).

Date: May 8, 2015.

2010 *Mathematics Subject Classification.* 20F36 (20F65, 57M25).

Key words and phrases. Configuration space; Braid group; Automorphism; Knots and links.

There is a 3-dimensional analogous of B_n: it is the fundamental group of all configurations of n unlinked Euclidean circles. Following [5] we will denote by $\mathcal{R}_n$ the space of configurations of n unlinked Euclidean circles and by R_n its fundamental group (called *group of rings* in [5]). The group R_n is generated by 3 types of moves (see figure 2).

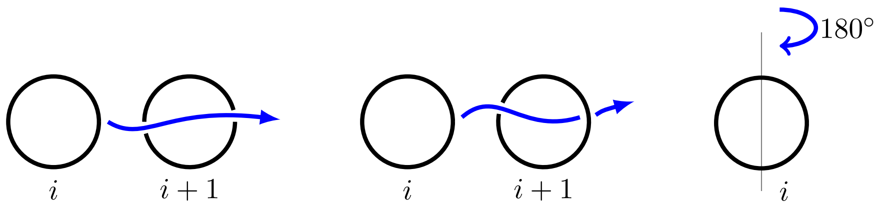

FIGURE 2. The move ρ_i (left). The move σ_i (center). The move τ_i (right).

The move ρ_i is the path permuting the i-th and the $i+1$-th circles by passing over (or around) while σ_i permutes them by passing the i-th circle through the $i+1$-th and τ_i is a 180° rotation of the circle back to itself, which verifies $\tau_i^2 = \mathrm{id}$ (see [5], with reverse notation for σ_i and ρ_i, see also [13]). To avoid the last move τ_i one can define $\mathcal{UR}_n$ as the configuration of n unlinked Euclidean circles being all parallel to a fixed plane, say the yz-plane (*untwisted rings*). The fundamental group of this configuration space is denoted in [5] by UR_n but we will denote it by WB_n and we shall call it *welded braid group*, since this is the most usual name for this group which appears in the literature in other different contexts such as motion groups ([14],[15]), ribbon tubes ([2]) automorphisms of free groups ([4, 12]) or collections of paths ([12]).

This article is devoted to the relationship between configuration spaces of points over an annulus and configuration spaces of Euclidean circles. We will introduce the configuration space of a special link, composed with Euclidean circles, called the *necklace* and denoted by $\mathcal{L}_n$ and we will consider induced representations as automorphisms of free groups.

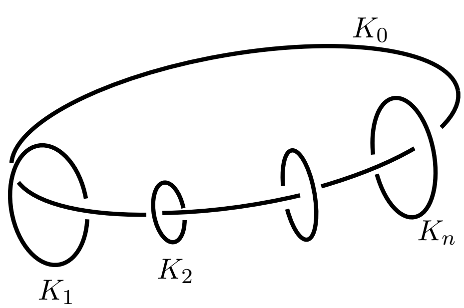

FIGURE 3. The necklace $\mathcal{L}_n$.

Our first result (Theorem 2) is that the fundamental group of the configuration space of n components necklaces (that we will call *braid group of a necklace*) is isomorphic to the fundamental group of the configuration space of n points over an annulus (which is CB_n) quotiented by the square of its center.

A theorem of Artin characterizes automorphisms of a free group coming from the action of the standard braid group. In our case, to a loop $\mathcal{L}_n(t)$ of necklaces we associate an automorphism from $\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n)$ to itself. Our second result, Theorem 9, is an analogous of Artin's theorem for the braid group of a necklace.

In section 5, we define affine braid groups of type $\mathcal{A}$ in terms of configurations of Euclidean circles and we refine the representation given in Theorem 9 to obtain a faithful representation and a characterization as automorphisms of free groups for affine braid groups of type $\mathcal{A}$ (Theorem 13): this is the third main result of the paper. In section 6 we show how to define the braid group B_n in terms of configurations of Euclidean circles and we give a short survey on some remarkable (pure) subgroups of WB_n; finally, in the Appendix we find the kernel of a particular representation of CB_n in Aut F_n, proving this way the statement of Theorem 1, which plays a key role in the proof of Theorem 13.

2. Necklaces and circular braids

2.1. The circular braid group. Recall that the circular braid group CB_n is the fundamental group of n distinct points in the plane less the origin (i.e. topologically an annulus). The circular braid group CB_n admits the following presentation (where the indices are defined modulo n, see for instance [17]):

$$CB_n = \left\langle \sigma_1, \ldots, \sigma_n, \zeta \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, 2 \ldots, n, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \neq 1, \\ \bar\zeta\sigma_i\zeta = \sigma_{i+1} & \text{for } i = 1, 2 \ldots, n \end{array} \right\rangle .$$

where $\bar\zeta$ stands for ζ^{-1}. Geometrically σ_i consists in permuting the i-th and $i+1$-th point and ζ in a cyclic permutation of the points.

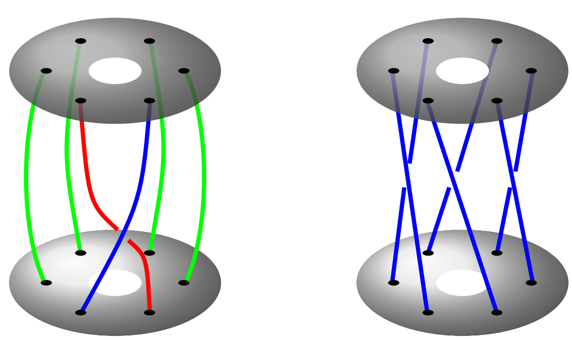

FIGURE 4. A move σ_i (left). The move ζ (right).

This is closed to a presentation of the classical braid group, with two major differences: (a) the indices are defined modulo n; (b) there are additional relations $\bar{\zeta}\sigma_i\zeta = \sigma_{i+1}$. In fact these latter relations enable to generate CB_n with only two generators: σ_1 and ζ. We will consider the (well defined) representation $\rho_{CB} : CB_n \to \operatorname{Aut} F_n$ defined as follows:

$$\rho_{CB}(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} \bar{x}_i \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \end{cases} \qquad \rho_{CB}(\zeta) : \{\ x_j \mapsto x_{j+1}$$

where indices are modulo n. The following Theorem will be proved in the Appendix and will play a key role in next sections.

Theorem 1. *The kernel of $\rho_{CB} : CB_n \to \operatorname{Aut} F_n$ is the cyclic group generated by ζ^n.*

2.2. **Necklaces.** Let $\mathcal{L}_n = K_0 \cup K_1 \cup \ldots \cup K_n$ be the following link :

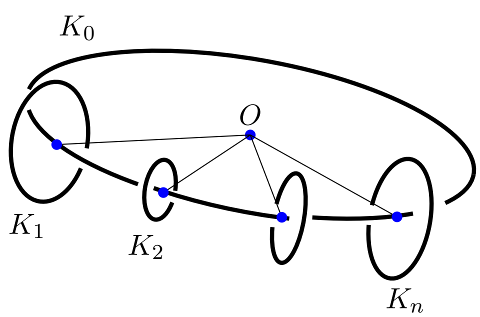

FIGURE 5. The necklace $\mathcal{L}_n$.

A link is called a *necklace* if:

- K_0 is an Euclidean circle of center O and of radius 1,
- each Euclidean circle K_i has a center O_i belonging to K_0, the circle K_i being of radius r_i with $0 < r_i < \frac{1}{2}$ and belonging to the plane containing the line (OO_i) and perpendicular to the plane of K_0,
- if $O_i = O_j$ then $r_i \neq r_j$.

We will suppose $n \geqslant 2$ and that the link is oriented.

In particular each K_i is a trivial knot such that:

$$\begin{cases} \operatorname{lk}(K_0, K_i) = +1 & i = 1, \ldots, n \\ \operatorname{lk}(K_i, K_j) = 0 & i, j = 1, \ldots, n, \quad i \neq j \end{cases}$$

2.3. **The iteration τ^n.** Let $\tau \in \pi_1(\operatorname{Conf} \mathcal{L}_n)$ denote the circular permutation of circles $K_1 \to K_2$, $K_2 \to K_3$,... It is important to understand the iteration τ^n. We give two visions of τ^n in $\pi_1(\operatorname{Conf} \mathcal{L}_n)$.

- This move can of course be seen as n iterations of τ: so that it is a full rotation of all the K_i ($i = 1, \dots, n$) along K_0, back to their initial position.
- τ^n can be seen in another way: suppose that $K_1, \dots, K_n$ are sufficiently closed circles. Fixing those K_i, τ^n corresponds to a full rotation of K_0 around all the K_i.

Indeed each of this move is rotation of angle 2π around an axis, and we can continuously change the axis to go from the first vision to the second.

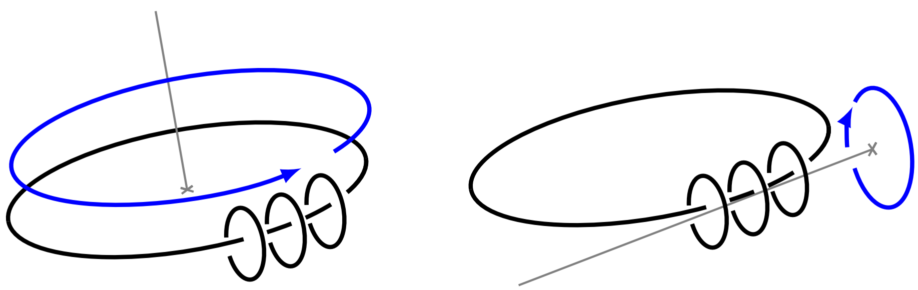

FIGURE 6. Two visions of τ^n.

2.4. **The fundamental group of** $\operatorname{Conf} \mathcal{L}_n$**.**

Theorem 2. *For $n \geqslant 2$, the group $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ is isomorphic to the circular braid group $CB_n/\langle \zeta^{2n} \rangle$.*

Proof. We start with a rigid configuration where K_0^* is the Euclidean circle in the plane (Oxy) centered at O and of radius 1. A necklace having such a core circle K_0^* is called a *normalized necklace* and we denote it by $\mathcal{L}_n^*$.

Lemma 3. $\operatorname{Conf} \mathcal{L}_n^*$ *is homeomorphic to the configuration space $\mathcal{CB}_n$.*

Proof of the lemma. We consider $\mathcal{CB}_n$ as the configuration space of n points lying on the annulus $A_0 = D_1 \setminus D_{\frac{1}{2}}$, where D_r denotes the closed disk in the plane (Oxy) centered at O of radius r. To a normalized necklace $\mathcal{L}_n^*$ we associate n points of A_0 as follows:

$$(K_1, \dots, K_n) \longmapsto (K_1 \cap A_0, \dots, K_n \cap A_0).$$

As each point of A_0 determines a unique normalized circle K_i, this map is a bijection. This maps and its inverse are continuous. □

We end the proof of Theorem 2 as follows. To a necklace $\mathcal{L}_n$ having core circle K_0 we associate a unit vector u_0, orthogonal to the plane containing K_0 (and oriented according to the orientation of K_0).

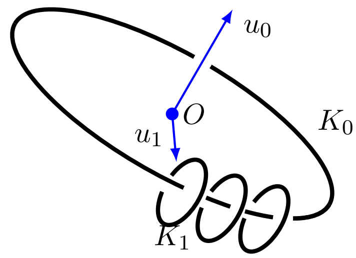

FIGURE 7. The normal vector u_0.

Let $G : \mathrm{Conf}(\mathcal{L}_n) \to S^2$ be the map defined by $G(\mathcal{L}_n) = u_0$. This map G is a locally trivial fibration whose fiber is $\mathrm{Conf}\,\mathcal{L}_n^*$ (for the unit vector $u_0 = (0,0,1)$). The long exact sequence in homotopy for the fibration $\mathrm{Conf}\,\mathcal{L}_n^* \hookrightarrow \mathrm{Conf}\,\mathcal{L}_n \twoheadrightarrow S^2$ provides:

$$0 \longrightarrow \pi_2(\mathrm{Conf}\,\mathcal{L}_n^*) \xrightarrow{H_2} \pi_2(\mathrm{Conf}\,\mathcal{L}_n) \xrightarrow{G_2} \pi_2(S^2)$$
$$\xrightarrow{d} \pi_1(\mathrm{Conf}\,\mathcal{L}_n^*) \xrightarrow{H_1} \pi_1(\mathrm{Conf}\,\mathcal{L}_n) \longrightarrow 0$$

It implies that H_1 is surjective (since in the exact sequence $\pi_1(S^2)$ is trivial).

Before computing the kernel of H_1, we give a motivation why H_1 is not injective. There is a natural map K from $\mathrm{Conf}\,\mathcal{L}_n$ to SO_3. Let us see SO_3 as the space of direct orthonormal frames (u_0, u_1, u_2). To a necklace $\mathcal{L}_n$ we associate u_0 as above, while u_1 is the unit vector from the origin O to K_1, then we set $u_2 = u_0 \wedge u_1$ (see figure 7).
Let us denote by $\pi : SO_3 \to S^2$, the natural projection $\pi(u_0, u_1, u_2) = u_0$. We have a commutative diagram, that is to say $G = \pi \circ K$. But as $\pi_2(SO_3) = 0$, it implies $G_2 = 0$. By the exact sequence, d is injective, so that $\pi_2(S^2) = \mathbb{Z} \cong \mathrm{Im}\, d = \mathrm{Ker}\, H_1$.
There is another interesting point with SO_3. In fact if we now see SO_3 as the space of rotations, we denote by ρ a full rotation around the vertical axis (supported by u_0). The $\rho \neq \mathrm{id}$, but $\rho^2 \simeq \mathrm{id}$ (because $\pi_1(SO_3) \cong \mathbb{Z}/2\mathbb{Z}$). For us the move ρ corresponds to the full rotation τ^n. It gives the idea that in $\mathrm{Conf}\,\mathcal{L}_n^*$, τ^n generates a subgroup isomorphic to $\mathbb{Z}$, but τ^{2n} is homotopic to id in $\mathrm{Conf}\,\mathcal{L}_n$.

We will now compute the kernel of H_1 as $\mathrm{Im}\, d$, where d is the boundary map coming from the exact sequence : $\mathrm{Ker}\, H_1 = \langle \tau^{2n} \rangle$.
We go back to the construction of this boundary map d (see for instance, [16, Theorem 6.12]) and we explicit one of the lifting. Let f be the generator of $\pi_2(S^2) \cong \mathbb{Z}$ defined by $f : S^2 \to S^2$, $f(x) = x$, but we prefer to see it as a map

$$f : I \times I \to S^2 \quad \text{such that} \quad f(\partial I^2) = N$$

where $N = (0,0,1)$ is the North pole of S^2.

We lift f to a map $\tilde{f}$ from $\partial I \times I \cup I \times \{0\}$ to the base-point $\mathcal{L}_n^*$ in Conf $\mathcal{L}_n$ (and Conf $\mathcal{L}_n^*$, for which u_0 is $\overrightarrow{ON}$).
By the homotopy lifting property, it extends to a map $h : I \times I \to \text{Conf}\,\mathcal{L}_n$, and $d(f) \in \pi_1(\text{Conf}\,\mathcal{L}_n^*)$ is the map induced by $h_{|I\times\{1\}}$ (figure 8).

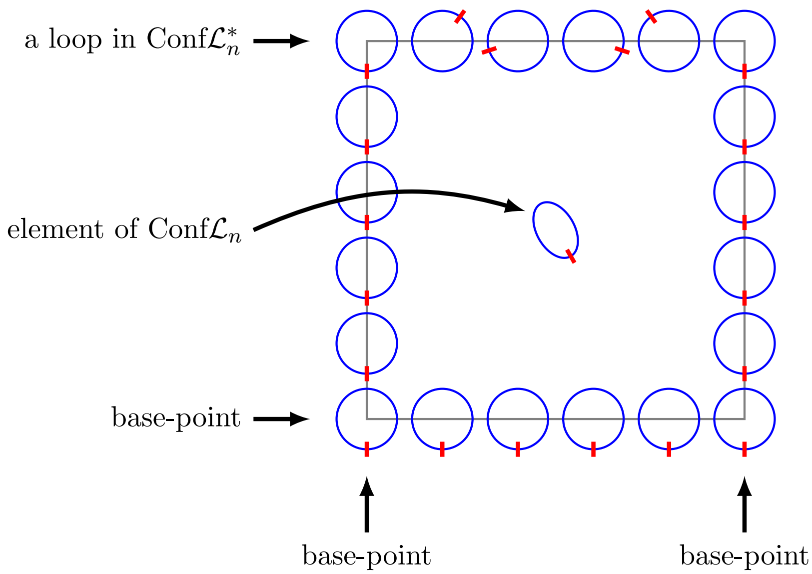

FIGURE 8. The lifting h from $I \times I$ to Conf $\mathcal{L}_n$ (K_0 in blue, K_1 in red).

One way to explicit this lifting is to see $I^2 \setminus \partial I^2$ as $S^2 \setminus \{N\}$. On S^2 there exists a continuous vector field, that is non-zero except at N (see the dipole integral curves in figure 10, see also [6, section 5] for other links between vector fields and configuration spaces). Now at each point $u_0 \in S^2 \setminus \{N\}$ is associated a non-zero vector u_1. Let us define $h : S^2 \setminus \{N\} \to \text{Conf}\,\mathcal{L}_n$ as follows: for $u_0 \in S^2 \setminus \{N\}$, $h(u_0)$ is the unique necklace $\mathcal{L}_n$, such $G(\mathcal{L}_n) = u_0$ and the unit vector from O to K_1 is u_1. We define $K_2, \ldots, K_n$ as parallel copies of K_1 (figure 9).

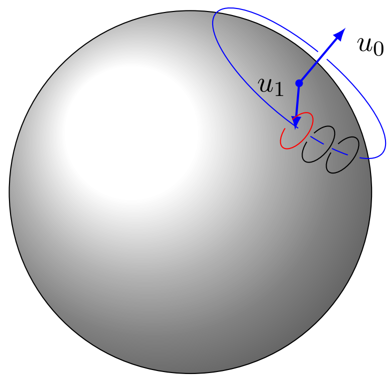

FIGURE 9. A necklace constructed from $u_0 \in S^2$ and $u_1 \in T_{u_0}S^2$.

Now $d(f)$ corresponds to the lift $h(\gamma)$ where γ is a small loop in S^2 around N. Due to the dipole figure at N, $h(\gamma)$ is exactly τ^{2n} (figures 10 and 11).

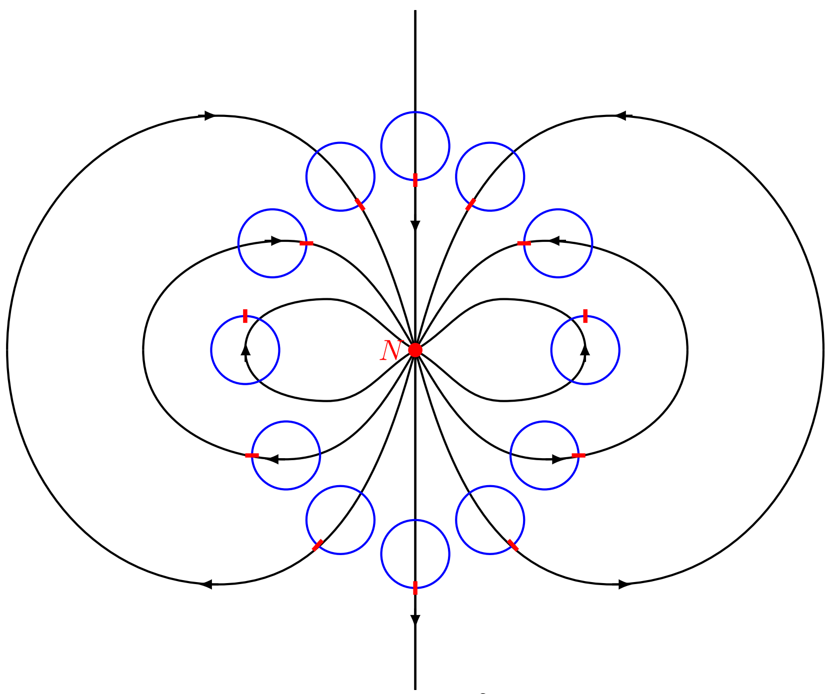

FIGURE 10. The vector fields on S^2 around N and the family of necklaces.

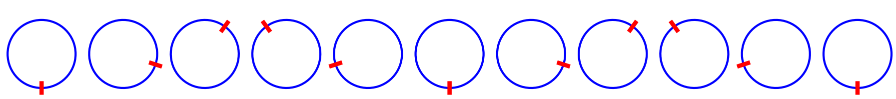

FIGURE 11. The family of necklaces τ^{2n}.

A first conclusion is that $\operatorname{Ker} H_1 = \langle d(f) \rangle = \langle \tau^{2n} \rangle$. And finally :

$$\pi_1(\operatorname{Conf} \mathcal{L}_n) \cong \pi_1(\operatorname{Conf} \mathcal{L}_n^*)/\langle \tau^{2n} \rangle \cong CB_n/\langle \zeta^{2n} \rangle$$

□

Let us end the section with few remarks:

Remark 4. Since ζ^n generates the center of CB_n then the group generated by ζ^{2n} is normal: so we can effectively write $\langle \zeta^{2n} \rangle$ instead of $\langle\langle \zeta^{2n} \rangle\rangle$; let us also recall that, denoting by $Mod_{n+2}(S^2)$ the mapping class group of the $n+2$-punctured sphere, $CB_n/\langle \zeta^n \rangle$ is isomorphic to the subgroup of $Mod_{n+2}(S^2)$ fixing two punctures (see [7]).

Remark 5. In Theorem 2 we constructed a map $h_n : CB_n \to \pi_1(\operatorname{Conf} \mathcal{L}_n)$ where the generator ζ in CB_n corresponds to the move τ in $\pi_1(\operatorname{Conf} \mathcal{L}_n)$, while the generator σ_i in CB_n corresponds to the move σ_i which permutes the i-th circle with the $i+1$-th one (modulo n) by passing the i-th circle through the $i+1$-th. The kernel of h_n is the group generated by ζ^{2n} and the group $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ admits the following group presentation:

$$\pi_1(\operatorname{Conf} \mathcal{L}_n) = \left\langle \sigma_1, \dots, \sigma_n, \tau \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, 2 \dots, n, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \neq 1, \\ \bar{\tau}\sigma_i\tau = \sigma_{i+1} & \text{for } i = 1, 2 \dots, n \\ \tau^{2n} = 1 & \end{array} \right\rangle.$$

3. Action on $\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n)$

As recalled in the introduction, we denote:

- the configuration space of n unlinked Euclidean circles by $\mathcal{R}_n$ and $\pi_1(\mathcal{R}_n)$ by R_n;
- the configuration of n unlinked Euclidean circles being all parallel to a fixed plane, say the yz-plane, by $\mathcal{UR}_n$ and its fundamental group by WB_n.

Clearly WB_n can be seen as a subgroup of R_n and it is generated by two families of elements, ρ_i and σ_i (see figure 2 and [5]): ρ_i is the path permuting the i-th and the $i+1$-th circles by passing over, while σ_i permutes them by passing the i-th circle through the $i+1$-th.

A *motion* of a compact submanifold N in a manifold M is a path f_t in $\operatorname{Homeo}_c(M)$ such that $f_0 = \operatorname{id}$ and $f_1(N) = N$, where $\operatorname{Homeo}_c(M)$ denotes the group of homeomorphisms of M with compact support. A motion is called *stationary* if $f_t(N) = N$ for all $t \in [0,1]$. The *motion group* $\mathcal{M}(M, N)$ of N in M is the group of equivalence classes of motion of N in M where two motions f_t, g_t are equivalent if $(g^{-1}f)_t$ is homotopic relative to endpoints to a stationary motion. The notion of motion groups was proposed by R. Fox, and studied by P. Dahm, one of his students. The first published article on the topic is [14]. Notice that motion groups generalize fundamental groups of configuration spaces, and that each motion is equivalent to a motion that fixes a point $* \in M \setminus N$, when M is non-compact, it is possible to define a homomorphism (the *Dahm morphism*):

$$D_N : \mathcal{M}(M,N) \to \mathrm{Aut}(\pi_1(M \setminus N, *))$$

sending an element represented by the motion f_t, into the automorphism induced on $\pi_1(M \setminus N, *)$ by f_1.
When $M = \mathbb{R}^3$ and N is a set $\mathcal{L}'$ of n unlinked Euclidean circles we get a map

$$D_{\mathcal{L}'} : R_n \to \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathcal{L}', *))$$

This map is injective (see [14]) and sends generators of R_n (and therefore of WB_n) in the following automorphisms of the free group $F_n = \langle x_1, \dots, x_n \rangle$:

$$\sigma_i : \begin{cases} x_i \mapsto x_i x_{i+1} \bar{x}_i \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \end{cases} \qquad \rho_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \end{cases}$$

$$\tau_j : \begin{cases} x_j \mapsto \bar{x}_j \\ x_k \mapsto x_k \quad k \neq j \end{cases}$$

where $i = 1, \dots, n-1$ and $j = 1, \dots, n$.
Now let $\mathcal{L}_n$ be a necklace: by forgetting the core circle K_0 we obtain a map from $\mathrm{Conf}\,\mathcal{L}_n$ to $\mathcal{R}_n$. To $\Gamma = \mathcal{L}_n(t)$ in $\pi_1(\mathrm{Conf}\,\mathcal{L}_n)$, we can therefore associate an automorphism $D_{\mathcal{L}_n}(\Gamma) : \pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n) \to \pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n)$. It is easy to compute the π_1 of the complement of $\mathcal{L}_n$ by giving its Wirtinger presentation:

$$\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n) = \langle x_1, \dots, x_n, y \mid x_i y = y x_i, i = 1 \dots, n \rangle$$

so that we obtain

$$\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n) \cong F_n \times \mathbb{Z}.$$

The action of the generators of $\pi_1(\mathrm{Conf}\,\mathcal{L}_n)$ are (with indices modulo n):

$$\sigma_i : \begin{cases} x_i \mapsto x_i x_{i+1} \bar{x}_i \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \\ y \mapsto y \end{cases} \qquad \tau : \begin{cases} x_j \mapsto x_{j+1} \\ y \mapsto y \end{cases}$$

Lemma 6. *(1) Let $D_{\mathcal{L}_n} : \pi_1(\mathrm{Conf}\,\mathcal{L}_n) \to \mathrm{Aut}\,\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n)$ be the Dahm morphism; then $\mathrm{Ker}\,D_{\mathcal{L}_n} = \langle \tau^n \rangle = \mathbb{Z}/2\mathbb{Z}$.*
(2) Let Φ be the natural map $\Phi : \pi_1(\mathrm{Conf}\,\mathcal{L}_n) \to R_n$ induced by forgetting the core circle K_0. Then $\mathrm{Ker}\,\Phi = \langle \tau^n \rangle = \mathbb{Z}/2\mathbb{Z}$.

Proof. We first notice some facts:

(a) The following diagram is commutative :

$$\begin{array}{ccc} \pi_1(\mathrm{Conf}\,\mathcal{L}_n) & \xrightarrow{D_{\mathcal{L}_n}} & \mathrm{Aut}\,\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n) \\ \Phi \downarrow & & \downarrow \Psi \\ R_n & \xrightarrow[D_{\mathcal{L}'_n}]{} & \mathrm{Aut}\,\pi_1(\mathbb{R}^3 \setminus \mathcal{L}'_n) \end{array}$$

where Φ and Ψ are natural maps induced by the inclusion $\mathcal{L}'_n \subset \mathcal{L}_n$. Remark that Ψ is then the map which forgets the generator y in $\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n)$ and therefore

$$\psi(D_{\mathcal{L}_n})(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} \bar{x}_i \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \end{cases} \qquad \psi(D_{\mathcal{L}_n})(\tau) : \{\ x_j \mapsto x_{j+1}$$

where indices are modulo n. Comparing with Theorem 1 and 2 we deduce that the $\Psi \circ D_{\mathcal{L}_n} \circ h_n = \rho_{CB}$ and $\mathrm{Ker}\,\Psi \circ D_{\mathcal{L}_n} \circ h_n = \langle \zeta^n \rangle$.
(b) As we already recall, it is known that for the trivial link $\mathcal{L}'_n$ the Dahm morphism $D_{\mathcal{L}'_n}$ is injective ([14]) (where $h_n : CB_n \to \pi_1(\mathrm{Conf}\,\mathcal{L}_n)$ is induced by 2).
(c) Ψ is injective when restricted to the image of $D_{\mathcal{L}_n}$. If $\Psi(f) = \mathrm{id}$, then $f(x_i) = x_i$ for all $i = 1, \ldots, n$. If $f \in \mathrm{Im}\, D_{\mathcal{L}_n}$, then due to the action of the generators (σ_i and τ), it implies $f(y) = y$. Finally if $\Psi(f) = \mathrm{id}$ and $f \in \mathrm{Im}\, D_{\mathcal{L}_n}$, then $f = \mathrm{id}$. Then $\mathrm{Ker}\, D_{\mathcal{L}_n} \circ h_n = \langle \zeta^n \rangle$. Clearly, $\tau^n \in \mathrm{Ker}\, D_{\mathcal{L}_n}$; on the other hand since h_n is surjective, if $x \in \mathrm{Ker}\, D_{\mathcal{L}_n}$, then $x \in \mathrm{Im}(\langle \zeta^n \rangle)$ and therefore $x \in \langle \tau^n \rangle$ (see Remark 5).

For the second statement it is therefore enough to prove that the kernel of $D_{\mathcal{L}_n}$ coincides with the kernel of Φ.

$$\begin{array}{llll} \gamma \in \mathrm{Ker}\,\Phi & \iff & \Phi(\gamma) = \mathrm{id} & \\ & \iff & D_{\mathcal{L}'_n} \circ \Phi(\gamma) = \mathrm{id} & \text{because } D_{\mathcal{L}'_n} \text{ is injective} \\ & \iff & \Psi \circ D_{\mathcal{L}_n}(\gamma) = \mathrm{id} & \text{because the diagram commutes} \\ & \iff & D_{\mathcal{L}_n}(\gamma) = \mathrm{id} & \text{because } \Psi_{|D_{\mathcal{L}_n}} \text{ is injective} \\ & \iff & \gamma \in \mathrm{Ker}\, D_{\mathcal{L}_n}(\gamma) & \end{array}$$

□

4. Characterization of automorphisms

We will use the following notation: for a word $\bar{w} = w^{-1}$, for an automorphism $\bar{\alpha} = \alpha^{-1}$. A famous result of Artin, characterizes automorphisms induced by braids.

Theorem 7 (Artin)**.** *The automorphisms induced by the action of B_n on F_n are exactly the automorphisms ϕ of $\mathrm{Aut}\, F_n$ that verify the two conditions below:*

(1) $$\phi(x_i) = w_i x_{\pi(i)} \bar{w}_i$$

for some $w_1, \dots, w_n \in F_n$ *and some permutation* $\pi \in \mathcal{S}_n$*, and:*

$$\phi(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n \tag{2}$$

Interestingly, if we do not require condition (2), we recover exactly automorphisms of $\operatorname{Aut} F_n$ induced by welded braids. Recall that the welded braid group is generated by two types of moves σ_i, ρ_i, which induced two kinds of automorphisms also denoted σ_i, ρ_i and described in section 3.

Theorem 8 (Theorem 4.1 of [12])**.** *The automorphisms of* $\operatorname{Aut} F_n$ *induced by the action of* WB_n *on* F_n *are exactly those verifying (1).*

As a straightforward consequence we have that the natural map $B_n \to WB_n$ sending σ_i into σ_i is injective. We will show in section 6 a geometric interpretation of such an embedding.

Now we relax condition (2) and characterize automorphisms induced by our configurations.

Theorem 9. *The automorphisms induced by the action* $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ *on* $\operatorname{Aut} F_n$ *are exactly the automorphisms* ϕ *of* $\operatorname{Aut} F_n$ *that verify the two conditions below:*

$$\phi(x_i) = w_i x_{\pi(i)} \bar{w}_i \tag{3}$$

for some $w_1, \dots, w_n \in F_n$ *and some permutation* $\pi \in \mathcal{S}_n$*, and:*

$$\phi(x_1 x_2 \cdots x_n) = w x_1 x_2 \cdots x_n \bar{w} \tag{4}$$

for some $w \in F_n$*.*

Proof. Notations:

- We denote by $\mathcal{A}_n$ the set of automorphisms of $\operatorname{Aut} F_n$ induced by $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ and $\mathcal{B}_n$ the set of automorphisms of $\operatorname{Aut} F_n$ that verify conditions (3) and (4). We will prove $\mathcal{A}_n = \mathcal{B}_n$.
- We set $\Delta = x_1 x_2 \cdots x_n$.
- And for any $w \in F_n$, we set the automorphism $g_w \in \operatorname{Aut} F_n$ defined by $g_w(x_i) = w x_i \bar{w}$. For any $w' \in F_n$, we have $g_w(w') = w w' \bar{w}$.

First of all, the action of $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ on $\operatorname{Aut} F_n$ is generated by the automorphisms σ_i $(i = 1, \dots, n)$ and τ that verify equations (3) and (4). In fact for $i = 1, \dots, n-1$, $\sigma_i(\Delta) = \Delta$; $\sigma_n(\Delta) = x_n \bar{x_1} \Delta x_1 \bar{x_n}$, $\tau(\Delta) = \bar{x_1} \Delta x_1$. It proves $\mathcal{A}_n \subset \mathcal{B}_n$.

The remaining part is to prove $\mathcal{B}_n \subset \mathcal{A}_n$: given an automorphism $f \in \operatorname{Aut} F_n$ that verifies conditions (3) and (4), we express it as the automorphism induced by some element of $\pi_1(\operatorname{Conf} \mathcal{L}_n)$.

1st step. Generation of g_{x_1}.

A simple verification prove the following equation: the automorphism g_{x_1} defined by $x_i \mapsto x_1 x_i \bar{x_1}$ is generated by elements of $\mathcal{A}_n$.

$$g_{x_1} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n-1} \circ \bar{\tau}$$

2nd step. Generation of g_{x_k}.
$g_{x_k} \in \mathcal{A}_n$ by the following relation:

$$g_{x_k} = \underbrace{\tau \circ \tau \circ \cdots \tau}_{k-1 \text{ occurrences}} \circ g_{x_1} \circ \underbrace{\bar{\tau} \circ \bar{\tau} \circ \cdots \bar{\tau}}_{k-1 \text{ occurrences}}$$

We also generate $g_{x_k^{-1}}$ as the inverse of g_{x_k}.

3rd step. Generation of g_w.
Let $w \in F_n$. We generate the automorphism g_w by induction on the length of w. Suppose that $w = x_k w'$ with $w' \in F_n$ of length strictly less than the length of w. Suppose that $g_{w'} \in \mathcal{A}_n$. Then

$$g_w = g_{x_k} \circ g_{w'} \in \mathcal{A}_n.$$

4th step. Simplification of the action on Δ.
Let $f \in \operatorname{Aut} F_n$. Suppose that f verifies conditions (3) and (4). In particular, let $w \in F_n$ such that $f(\Delta) = w\Delta\bar{w}$. Then $g_{\bar{w}} \circ f$ still satisfies conditions of type (3) and the condition (2): $g_{\bar{w}} \circ f(\Delta) = \Delta$.

5th step. Artin's theorem.
Therefore we can suppose that, given $f \in \mathcal{B}_n$ and after a composition $g \in \mathcal{A}_n$ of elements σ_i, τ, $g \circ f$ verifies condition (1) (which is exactly condition (3)) and condition (2). By Artin's theorem $g \circ f \in \mathcal{A}_n$. Hence $f \in \mathcal{A}_n$. It ends the proof of $\mathcal{B}_n \subset \mathcal{A}_n$, so that we get $\mathcal{A}_n = \mathcal{B}_n$.

□

5. Zero angular sum

We say that $\Gamma \in \pi_1(\operatorname{Conf} \mathcal{L}_n)$ has *zero angular sum* if $\Gamma \in \langle \sigma_1, \ldots, \sigma_n \rangle$. This definition is motivated by the fact that a move σ_i shifts the component K_i by an angle of –say– $+\frac{2\pi}{n}$ while K_{i+1} is shifted by $-\frac{2\pi}{n}$, the sum of these angles being zero. On the other hand τ, moves each K_i by an angle of –say– $+\frac{2\pi}{n}$, with a sum of 2π. The aim of this section is to characterize the zero angular sum condition at the level of automorphisms. We will define a kind of total winding number $\epsilon(\Gamma)$ about the axis of rotation of K_0.
Let $\epsilon : \pi_1(\operatorname{Conf} \mathcal{L}_n) \to \mathbb{Z}$ defined as follows: to $\Gamma \in \pi_1(\operatorname{Conf} \mathcal{L}_n)$, we associate an automorphism $\phi = D_{\mathcal{L}_n}(\Gamma)$ by the Dahm morphism. By theorem 9, $\phi(x_1 x_2 \cdots x_n) = w x_1 x_2 \cdots x_n \bar{w}$, for some $w \in F_n$. We define $\epsilon(\Gamma) = \ell(\bar{w}) \in \mathbb{Z}$ to be the algebraic length of the word $\bar{w}$. We have the following characterization of zero angular sum:

Proposition 10. *$\Gamma \in \langle \sigma_1, \ldots, \sigma_n \rangle$ if and only if $\epsilon(\Gamma) = 0$.*

Proof. We have the following facts:

- ϵ is a morphism.
- If we denote $\Delta = x_1 \cdots x_n$ then for $i = 1, \ldots, n-1$, $\sigma_i(\Delta) = \Delta$; $\sigma_n(\Delta) = x_n \bar{x_1} \Delta x_1 \bar{x_n}$, $\tau(\Delta) = \bar{x_1} \Delta x_1$. It implies $\epsilon(\sigma_i) = 0$ and $\epsilon(\tau) = 1$.
- Any $\Gamma \in \pi_1(\operatorname{Conf} \mathcal{L}_n)$ can be written $\Gamma = \tau^k \sigma_{i_1} \cdots \sigma_{i_\ell}$ (by using the relations $\sigma_i \tau = \tau \sigma_{i+1}$).
- Hence $\epsilon(\Gamma) = 0$ implies $k = 0$, in which case $\Gamma \in \langle \sigma_1, \ldots, \sigma_n \rangle$.

□

We recall that the *affine braid group* $\tilde{A}_{n-1}$, is the group obtained by the group presentation of B_{n+1} by replacing the relation $\sigma_n \sigma_1 = \sigma_1 \sigma_n$ with the relation $\sigma_n \sigma_1 \sigma_n = \sigma_1 \sigma_n \sigma_1$. By comparison of group presentations one deduces that $CB_n = \tilde{A}_{n-1} \rtimes \langle \zeta \rangle$ (see also [7, 17]). It then follows from Theorem 2 and Remark 5 that $\pi_1(\operatorname{Conf} \mathcal{L}_n) = \tilde{A}_{n-1} \rtimes \langle \tau \rangle$. Since τ has finite order, $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ inherits some properties of $\tilde{A}_{n-1}$, in particular:

Corollary 11. *The group $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ is linear and, provided with the group presentation given in Remark 5, has solvable word problem.*

On the other hand, it follows from Theorem 2 and Proposition 10 that:

Proposition 12. *The affine braid group on n strands, $\tilde{A}_{n-1}$ is isomorphic to the subgroup of $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ consisting of elements of zero angular sum.*

Consider now the representation $\rho_{\mathrm{Aff}} : \tilde{A}_{n-1} \longrightarrow \operatorname{Aut} F_n$ induced by the action of $\pi_1(\operatorname{Conf} \mathcal{L}_n)$ on $\operatorname{Aut} F_n$ (obtained by setting $y = 1$).

$$i \neq n : \rho_{\mathrm{Aff}}(\sigma_i) : \begin{cases} x_i \longmapsto x_i \, x_{i+1} \, \bar{x}_i, & \\ x_{i+1} \longmapsto x_i, & \\ x_j \longmapsto x_j, & j \neq i, i+1. \end{cases}$$

$$\rho_{\mathrm{Aff}}(\sigma_n) : \begin{cases} x_1 \longmapsto x_n & \\ x_n \longmapsto x_n x_1 \bar{x_n}, & \\ x_j \longmapsto x_j, & j \neq 1, n. \end{cases}$$

Theorem 13. *i) The representation $\rho_{\mathrm{Aff}} : \tilde{A}_{n-1} \longrightarrow \operatorname{Aut} F_n$ is faithful.*
ii) An element $\phi \in AutF_n$ belongs to $\rho_{\mathrm{Aff}}(\tilde{A}_{n-1})$ if and only if it verifies the conditions below:

$$\phi(x_i) = w_i x_{\pi(i)} \bar{w}_i \tag{5}$$

for some $w_1, \ldots, w_n \in F_n$ and some permutation $\pi \in \mathcal{S}_n$, and:

$$\phi(x_1 x_2 \cdots x_n) = w x_1 x_2 \cdots x_n \bar{w} \tag{6}$$

for some $w \in F_n$ with algebraic length $\ell(w) = 0$.

Proof. The kernel of ρ_{Aff} is a subgroup of the kernel of $\Psi \circ D_{\mathcal{L}_n} \circ h_n$, which is generated by τ^n (Lemma 6). Since τ^n generates the center of $\pi_1(\operatorname{Conf} \mathcal{L}_n)$, the kernel of ρ_{Aff} is a subgroup of the center of $\tilde{A}_{n-1}$. Since the center of $\tilde{A}_{n-1}$ is trivial (see for instance [1]) we can conclude that ρ_{Aff} is faithful. The characterization given in the second statement follows combining Theorem 9 and Proposition 10. □

6. The linear necklace

Let $\mathcal{C}_n^* = K_1 \cup \ldots \cup K_n$ be the link where each K_i is a Euclidean circle parallel to the yz-plane and centered at the x-axis. We call such a link a *linear necklace*, thinking of the x-axis as K_0, a circle passing through a point at infinity.

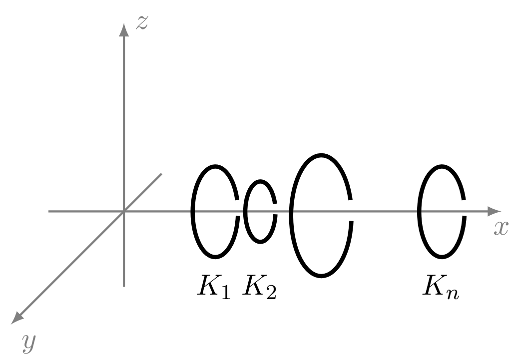

Figure 12. A linear necklace.

We recall that $\mathcal{UR}_n$ is the configuration space of n disjoint Euclidean circles lying on planes parallel to the yz-plane. We have that:

Theorem 14. *The inclusion of* $\operatorname{Conf}\mathcal{C}_n^*$ *into* $\mathcal{UR}_n$ *induces an injection at the level of fundamental groups.*

Proof. Let us remark that the moves $\sigma_1, \ldots, \sigma_{n-1}$ depicted in figure 2 belong to $\pi_1(\operatorname{Conf}\mathcal{C}_n^*)$. Actually, they generate $\pi_1(\operatorname{Conf}\mathcal{C}_n^*)$; in fact, the position of any circle K_i of $\mathcal{C}_n^*$ is determined by the intersection with the half-plane $y = 0$ and $z > 0$. It follows that the configuration space of linear necklaces, $\operatorname{Conf}\mathcal{C}_n^*$, can be identified with the configuration space of n distinct points in the (half-)plane, so that $\pi_1(\operatorname{Conf}\mathcal{C}_n^*) = B_n$, where generators are exactly moves $\sigma_1, \ldots, \sigma_{n-1}$. □

The previous result provides then a geometrical interpretation for the algebraic embedding of B_n into WB_n as subgroups of $\operatorname{Aut}(F_n)$.

Pure subgroups. Let us denote by $\operatorname{Conf}_{\mathrm{Ord}}\mathcal{C}_n$ the configuration space of n disjoint Euclidean ordered circles lying on planes parallel to the yz-plane: $\pi_1(\operatorname{Conf}_{\mathrm{Ord}}\mathcal{C}_n)$ is called the *pure welded braid group* on n strands and will be denoted by WP_n, while $\pi_1(\operatorname{Conf}_{\mathrm{Ord}}\mathcal{C}_n^*)$ is isomorphic to the pure braid group P_n. Previous results imply that P_n embeds geometrically in WP_n.
More precisely, the group $\pi_1(\operatorname{Conf}_{\mathrm{Ord}}\mathcal{C}_n^*)$ is generated by the family of paths $\lambda_{i,j}$ for $1 \leqslant i < j \leqslant n$: $\lambda_{i,j}$ moves only the i-th circle that passes inside the

following ones until the $j-1$-th one, inside-outside the j-th one and that finally comes back passing inside the other circles.
Notice also that in [5] was introduced the configuration spaces of circles lying on parallel planes of different size, that we can denote by $\mathcal{UR}_n^<$. We can take as base-point for $\pi_1(\mathcal{UR}_n^<)$ a configuration of parallel circles with center on the z-axis and such that for any $i=1,\dots,n-1$ the i-th circle has radius greater than the radius of the $i+1$-th one. Let us remark that all other choices of base point give conjugated subgroups in WP_n corresponding to different permutations of circles. As shown in [5], $\pi_1(\mathcal{UR}_n^<)$ is generated by $\delta_{i,j}$ for $1 \leqslant i < j \leqslant n$: $\delta_{i,j}$ moves only the i-th circle that passes outside the $j-1$-th one (without passing inside-outside the other circles) and moves back (without passing inside-outside the other circles).
Let us recall that $\pi_1(\mathcal{UR}_n^<)$ is called *upper McCool group* in [10] and denoted by $P\Sigma_n^+$: it is interesting to remark that $P\Sigma_n^+$ and P_n have isomorphic Lie algebras associated to the lower central series and the groups themselves are isomorphic for $n=2,3$ ([10]). A. Suciu communicated to us that using ideas from [9] it is possible to show that the ranks of *Chen groups* are different for $n>3$ and therefore $P\Sigma_n^+$ and P_n are not isomorphic for $n>3$.

Appendix: proof of Theorem 1

Theorem 1. *The kernel of $\rho_{CB}: CB_n \to \operatorname{Aut} F_n$ is the cyclic group generated by ζ^n.*

- Let us recall that when we restrict the map $D_{\mathcal{L}'}: R_n \to \operatorname{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathcal{L}', *))$ to the braid group B_n we get the usual Artin representation $\rho_A : B_n \to \operatorname{Aut} F_n$:

$$\rho_A(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} \bar{x}_i \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad j \neq i, i+1 \end{cases}$$

  for the usual generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n.

  Therefore we can define the group $B_n \ltimes_{\rho_A} F_n$: as generators we will denoted by $\alpha_1, \dots, \alpha_{n-1}$ the generators of the factor B_n and by $\eta_1, \dots \eta_n$ a set of generators for F_n. According to such a set of generators a possible complete set of relations is the following:

$$\begin{array}{l} \alpha_i\alpha_{i+1}\alpha_i = \alpha_{i+1}\alpha_i\alpha_{i+1} \quad \text{for } i=1,2\dots,n-1, \\ \alpha_i\alpha_j = \alpha_j\alpha_i \quad \text{for } |i-j| \neq 1, \\ \alpha_i^{-1}\eta_i\alpha_i = \eta_i\eta_{i+1}\eta_i^{-1} \quad \text{for } i=1,2\dots,n-1 \\ \alpha_i^{-1}\eta_{i+1}\alpha_i = \eta_i \quad \text{for } i=1,2\dots,n-1 \\ \alpha_i^{-1}\eta_k\alpha_i = \eta_k \quad \text{for } i=1,2\dots,n-1 \quad \text{and } k \neq i, i+1 \end{array}$$

- The group CB_n is isomorphic to $B_n \ltimes_{\rho_A} F_n$ ([11]); we left to the reader the verification that a possible isomorphism is the map $\Theta_n : CB_n \to B_n \ltimes_{\rho_A} F_n$ defined as follows: $\Theta_n(\zeta) = \sigma_{n-1}\cdots\sigma_1\eta_1$, $\Theta_n(\sigma_j) = \alpha_j$ for $j=1,\dots,n-1$ and $\Theta_n(\sigma_n) = \eta_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\eta_1$.

Using the action by conjugation of F_n on itself we get a representation $\chi_n : B_n \ltimes_{\rho_A} F_n \to \operatorname{Aut} F_n$. More precisely, $\chi_n(\alpha_j) = \rho_A(\sigma_j)$ and $\chi_n(\eta_i)(x_k) = x_i^{-1} x_k x_i$ for any $j = 1, \ldots, n-1$ and $i, k = 1, \ldots, n$.

- One can easily verify on the images of generators that the composed homomorphism $\chi_n \circ \Theta_n : CB_n \to \operatorname{Aut} F_n$ coincides with $\rho_{CB} : CB_n \to \operatorname{Aut} F_n$ defined in Section 2. We claim that the kernel of χ_n is generated by $\Theta(\zeta^n)$: since Θ_n is an isomorphism, then the kernel of ρ_{CB} is generated by ζ^n.

  To prove that, remark that the group generated by $\Theta(\zeta^n)$ is in the kernel of χ_n ; let $w \in \operatorname{Ker} \chi_n$ and write w in the form $w = \alpha\, \eta$ where α is written in the generators α_i's and η in the generators η_j's.

  Since $\chi_n(w)(x_j) = x_j$ for all generators $x_1, \ldots, x_n$ of F_n, we have that $\chi_n(\alpha)(x_j) = \eta^{-1}(x_j)$, where $\chi_n(\alpha)(x_j) = \rho_A(\alpha)(x_j)$, identifying any usual braid generator σ_i with corresponding α_i.

  It follows that $\rho_A(\alpha)$ is an inner automorphism, therefore α belongs to the center of B_n (see for instance [3], Remark 1): more precisely $\alpha = ((\alpha_{n-1} \cdots \alpha_1)^n)^m$ for some $m \in \mathbb{Z}$ and $\rho_A(\alpha)(x_j) = (x_1 \cdots x_n)^m x_j (x_1 \cdots x_n)^{-m}$. Then we can deduce that $\eta = (\eta_1 \cdots \eta_n)^m$ and $w = ((\alpha_{n-1} \cdots \alpha_1)^n)^m (\eta_1 \cdots \eta_n)^m$. Using the defining relations of $B_n \ltimes_{\rho_A} F_n$ we obtain that $w = (((\alpha_{n-1} \cdots \alpha_1)\eta_1)^n)^m = \Theta(\zeta^n)^m$.

Acknowledgments

The authors thank the referee of a previous version for his comments and Juan González-Meneses for useful discussions on representations of affine braid groups in terms of automorphisms. The authors are deeply indebted to Martin Palmer for fruitful conversations on configurations spaces. The research of the first author was partially supported by the French grant ANR-11-JS01-002-01 "VASKHO". The research of the second author was partially supported by the French grant ANR-12-JS01-002-01 "SUSI".

References

[1] M. A. Albar and G. L. Johnson, *The centre of the circular braid group.* Math. Japonica 30 (1985) 641–645.

[2] B. Audoux, P. Bellingeri, J.-B. Meilhan, E. Wagner, *Homotopy classification of ribbon tubes and welded string links.* Preprint, arXiv:1407.0184.

[3] V. Bardakov and P. Bellingeri, *On representations of Artin-Tits and surface braid groups.* J. Group Theory 14 (2011) 143–163.

[4] B. Berceanu, S. Papadima, *Universal representations of braid and braid-permutation groups.* J. Knot Theory Ramifications 18 (2009) 999–1019.

[5] T. Brendle, A. Hatcher, *Configuration spaces of rings and wickets.* Comment. Math. Helv. 88 (2013) 131–162.

[6] F. Cantero, M. Palmer, *On homological stability for configuration spaces on closed background manifolds.* Preprint, arXiv:1406.4916.

[7] R. Charney, J. Crisp, *Automorphism groups of some affine and finite type Artin groups.* Math. Res. Lett. 12 (2005) 321–333.

[8] R. Charney, D. Peifer, *The $K(\pi, 1)$ conjecture for affine braid groups.* Comment. Math. Helv. 78 (2003) 584–600.

[9] D. C. Cohen, A. I. Suciu, *The Chen groups of the pure braid group.* The Čech centennial (Boston, MA, 1993), 45–64, Contemp. Math., 181, Amer. Math. Soc., Providence, RI, 1995.
[10] F. R. Cohen, J. Pakianathan, V. V. Vershinin, J. Wu, *Basis-conjugating automorphisms of a free group and associated Lie algebras.* Groups, homotopy and configuration spaces, 147–168, Geom. Topol. Monogr., 13, Geom. Topol. Publ., Coventry, 2008.
[11] J. Crisp and L. Paris, *Artin groups of type B and D.* Adv. Geom. 5 (2005), 607–636.
[12] R. Fenn, R. Rimányi, C. Rourke, *The braid-permutation group.* Topology 36 (1997) 123–135.
[13] M. Freedman, R. Skora, *Strange actions of groups on spheres.* J. Differential Geom. 25 (1987) 75–98.
[14] D. Goldsmith, *The theory of motion groups.* Michigan Math. J. 28 (1981) 3–17.
[15] D. Goldsmith, *Motion of links in the 3-sphere.* Math. Scand. 50 (1982) 167–205.
[16] M. Hutchings, *Introduction to higher homotopy groups and obstruction theory.* Link to personal web page.
[17] R. Kent, D. Peifer, *A geometric and algebraic description of annular braid groups.* Internat. J. Algebra Comput. 12 (2002) 85–97.

E-mail address: `Paolo.Bellingeri@unicaen.fr`

E-mail address: `Arnaud.Bodin@math.univ-lille1.fr`

Laboratoire Nicolas Oresme, Université de Caen, 14032 Caen, France

Laboratoire Paul Painlevé, Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq, France