MILNOR FIBRATIONS OF MEROMORPHIC FUNCTIONS

ARNAUD BODIN, ANNE PICHON, JOSÉ SEADE

Abstract. In analogy with the holomorphic case, we compare the topology of Milnor fibrations associated to a meromorphic germ f/g : the local Milnor fibrations given on Milnor tubes over punctured discs around the critical values of f/g, and the Milnor fibration on a sphere.

1. Introduction

The classical fibration theorem of Milnor in [6] says that every holomorphic map (germ) $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $n \geqslant 2$ and a critical point at $0 \in \mathbb{C}^n$ has two naturally associated fibre bundles, and both of these are equivalent. The first is:

$$\phi = \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \longrightarrow \mathbb{S}^1 \tag{1}$$

where $\mathbb{S}_\varepsilon$ is a sufficiently small sphere around $0 \in \mathbb{C}^n$ and $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link of f at 0. The second fibration is:

$$f : \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1 \tag{2}$$

where $\mathbb{B}_\varepsilon$ is the closed ball in $\mathbb{C}^n$ with boundary $\mathbb{S}_\varepsilon$ and $\mathbb{D}_\delta$ is a disc around $0 \in \mathbb{C}$ which is sufficiently small with respect to ε.

The set $N(\varepsilon, \delta) = \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\delta)$ is usually called a local *Milnor tube* for f at 0, and it is diffeomorphic to $\mathbb{S}_\varepsilon$ minus an open regular neighbourhood T of K. (Thus, to get the equivalence of the two fibrations one has to "extend" the latter fibration to $T \setminus K$.) In fact, in order to have the second fibration one needs to know that every map-germ f as above has the so-called "Thom property", which was not known when Milnor wrote his book. What he proves is that the fibers in (1) are diffeomorphic to the intersection $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ for t close enough to 0. The statement that (2) is a fibre bundle was proved later in [5] by Lê

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Dũng Tráng in the more general setting of holomorphic maps defined on arbitrary complex analytic spaces, and we call it the *Milnor-Lê* fibration of f. Once we know that (2) is a fibre bundle, the arguments of [6, Chapter 5] show this is equivalent to the *Milnor* fibration (1).

The literature about these fibrations is vast, and so are their generalizations to various settings, including real analytic map-germs and meromorphic maps, and that is the starting point of this article.

Let U be an open neighbourhood of 0 in $\mathbb{C}^n$ and let $f, g : U \longrightarrow \mathbb{C}$ be two holomorphic functions without common factors such that $f(0) = g(0) = 0$. Let us consider the meromorphic function $F = f/g : U \to \mathbb{C}P^1$ defined by $(f/g)(x) = [f(x)/g(x)]$. As in [3], two such germs at 0, $F = f/g$ and $F' = f'/g'$ are considered as equal (or equivalent) if and only if $f = hf'$ and $g = hg'$ for some holomorphic germ $h : \mathbb{C}^n \to \mathbb{C}$ such that $h(0) \neq 0$. Notice that f/g is not defined on the whole U ; its indetermination locus is

$$I = \Big\{ z \in U \mid f(x) = 0 \text{ and } g(x) = 0 \Big\} .$$

In particular, the fibers of $F = f/g$ do not contain any point of I : for each $c \in \mathbb{C}$, the fiber $F^{-1}(c)$ is the set

$$F^{-1}(c) = \Big\{ x \in U \mid f(x) - cg(x) = 0 \Big\} \setminus I .$$

In a series of articles, S. M. Gusein-Zade, I. Luengo and A. Melle-Hernández, and later D. Siersma and M. Tibăr, studied local Milnor fibrations of the type (2) associated to every critical value of the meromorphic map $F = f/g$. See for instance [3, 4], or Tibar's book [12] and the references in it. Of course the "Milnor tubes" $\mathbb{B}_\varepsilon \cap F^{-1}(\partial \mathbb{D}_\delta)$ in this case are not actual tubes in general, since they may contain $0 \in U$ in their closure. These are in fact "pinched tubes".

It is thus natural to ask whether one has for meromorphic map-germs fibrations of Milnor type (1), and if so, how these are related to those of the Milnor-Lê type (2) studied (for instance) in [3, 4, 12]. The first of these questions was addressed in [10, 1, 11] from two different viewpoints, while the answer to the second question is the bulk of this article.

In fact, it is proved in [1] that if the meromorphic germ $F = f/g$ is semitame (see the definition in Section 2), then

$$\frac{F}{|F|} = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1 \tag{3}$$

is a fiber bundle, where $L_f = \{f = 0\} \cap \mathbb{S}_\varepsilon$ and $L_g = \{g = 0\} \cap \mathbb{S}_\varepsilon$ are the oriented links of f and g. Notice that away from the link $L_f \cup L_g$

one has an equality of maps:

$$\frac{f/g}{|f/g|} = \frac{f\bar{g}}{|f\bar{g}|},$$

where $\bar{g}$ denotes complex conjugation. It is proved in [11] that if the real analytic map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$ and satisfies the Thom property, then the Milnor-Lê fibration of $f\bar{g}$,

$$N(\varepsilon,\delta) := \mathbb{B}_\varepsilon \cap (f\bar{g})^{-1}(\partial\mathbb{D}_\delta) \xrightarrow{f\bar{g}} \partial\mathbb{D}_\delta \cong \mathbb{S}^1, \tag{4}$$

is equivalent to the Milnor fibration (3) of f/g when this map is semitame. That is, the fibration (4) on the Milnor tube $N(\varepsilon,\delta)$ of $f\bar{g}$ is equivalent to the Milnor fibration (3) of the meromorphic germ f/g.

In this article we complete the picture by comparing the local fibrations of Milnor-Lê type of a meromorphic germ f/g studied by Gusein-Zade *et al*, with the Milnor fibration (3). We prove that if the germ f/g is semitame and (IND)-tame (see Sections 2 and 3), then the global Milnor fibration (3) for f/g is obtained from the local Milnor fibrations of f at 0 and ∞ by a gluing process that is, fiberwise, reminiscent of the classical connected sum of manifolds (see Theorem 9, and its corollaries, in Section 5).

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2. Semitameness and the global Milnor fibration of F

Adapting Milnor's definition [6], we define the gradient of $F = f/g$ at a point $x \in U \setminus I$ by :

$$\operatorname{grad}(f/g) = \left(\overline{\frac{\partial(f/g)}{\partial x_1}}, \dots, \overline{\frac{\partial(f/g)}{\partial x_n}} \right).$$

The following definitions were introduced in [1] following ideas of [7]. We consider the set

$$M(F) = \big\{ x \in U \setminus I \mid \exists \lambda \in \mathbb{C}, \operatorname{grad}(f/g)(x) = \lambda x \big\}$$

consisting of the points of non-transversality between the fibres of f/g and the spheres $\mathbb{S}_r$ centered at the origin of $\mathbb{C}^n$.

Definition 1. We define a bifurcation set B for the meromorphic function $F = f/g$ as follows. A value $c \in \mathbb{C}P^1$ is in B if and only if there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $M(F)$ such that

$$\lim_{k\to\infty} x_k = 0 \quad \text{and} \quad \lim_{k\to\infty} F(x_k) = c\,.$$

Later, in Section 4, we will compare this bifurcation set B with the set of *atypical values* of F introduced in [3].

Let $L_f = \{f = 0\} \cap \mathbb{S}_\varepsilon$ and $L_g = \{g = 0\} \cap \mathbb{S}_\varepsilon$ be the oriented links of f and g.

Let W be a subset of $\mathbb{C}P^1$ and consider the map

$$\Phi_W = \frac{f/g}{|f/g|} : \big(\mathbb{S}_\varepsilon \setminus (L_f \cup L_g)\big) \cap F^{-1}(W) \longrightarrow \mathbb{S}^1\,.$$

Proposition 2. *If W is an open set in $\mathbb{C}P^1$ such that $W \cap B = \emptyset$, then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \leqslant \varepsilon_0$, the map Φ_W is a C^∞ locally trivial fibration over its image.*

The proof is that of [1, Theorem 2.6]; it follows Milnor's proof [6, Chapter 4] with minor modifications. See also [7]. The main modification of Milnor's proof concerns Lemma 4.4 of [6], for which an adapted formulation and a detailed proof is given in [1, Lemma 2.7].

Definition 3. The meromorphic function f/g is *semitame* at 0 if $B \subset \{0, \infty\}$.

Proposition 2 is a more general statement than [1, Theorem 2.6]. When F is semitame, the following is obtained by applying Proposition 2 to $W = \mathbb{C}P^1 \setminus \{0, \infty\}$:

Corollary 4. *(*[1, Theorem 2.6]*) If F is semitame, then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \leqslant \varepsilon_0$, the map*

$$\Phi_F = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1$$

is a C^∞ locally trivial fibration.

Definition 5. When F is semitame, we call Φ_F *the global Milnor fibration* of the meromorphic germ F. We denote by $\mathcal{M}_F$ the fibre of Φ_F and we call it the *global Milnor fibre* of F.

It is shown in [11] that Φ_F is a fibration of the multilink $L_f \cup -L_g$, where $-L_g$ means L_g with the opposite orientation.

For our purpose, it will be necessary to consider the restriction $\check{\Phi}_F$ of Φ_F to $\big(\mathbb{S}_\varepsilon \setminus (L_f \cup L_g)\big) \cap F^{-1}(\mathbb{D}_R(0) \setminus \mathbb{D}_\delta(0))$ where $\delta \ll 1$ and $1 \ll R$.

Definition 6. We denote by $\check{\mathcal{M}}_F$ the fibre of $\check{\Phi}_F$ and we call it the *truncated global Milnor fibre* of F.

3. Tameness near the indetermination points

In this section we introduce a technical condition on f/g: the (IND)-tameness ((IND) for "indetermination") which enables us to control the behaviour of f/g in a neighbourhood of its indetermination points when $n \geqslant 3$. This condition will appear as an essential hypothesis for our main Theorem 9. Note that this section only concerns the case $n \geqslant 3$.

Let us fix $r > 0$ and let us consider some sufficiently small constants $0 < \varepsilon' \ll \delta \ll \varepsilon \ll 1$. These constants will be defined more precisely in the proof of Theorem 9.

Let $X = F^{-1}\left(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)\right) \cap \left(\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}\right)$. See Figure 2 in Section 7.

For $\eta > 0$, we consider the neighbourhood of I defined by:

$$N_\eta = \left\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 \leqslant \eta^2\right\},$$

and its boundary,

$$\partial N_\eta = \left\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 = \eta^2\right\}.$$

The proof of Theorem 9 is based on the existence of a vector field v on X which satisfies for all sufficiently small η, $0 < \eta \ll \varepsilon'$ the following properties (see Figure 3 in Section 7):

- **(1)** The argument of F is constant along the integral curves of v.
- **(2)** The norm of z is strictly increasing along the integral curves of v.
- **(3)** For all $z \in N_\eta$, the integral curve passing through z is contained in the tube $\partial N_{\eta'}$ where ${\eta'}^2 = |f(z)|^2 + |g(z)|^2$.

In this paper, we use two different inner products on $\mathbb{C}^n$:

(HF) The usual hermitian form $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined for $z = (z_1, \ldots, z_n), z' = (z'_1, \ldots, z'_n) \in \mathbb{C}^n$ by:

$$\langle z, z' \rangle = \sum_{k=1}^{n} z_k \bar{z}'_k.$$

(IP) The usual inner product $\langle\ ,\ \rangle_{\mathbb{R}} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ on $\mathbb{R}^{2n}$:

$$\langle z, z' \rangle_{\mathbb{R}} = \sum_{k=1}^{n} (x_k x'_k + y_k y'_k),$$

where for all k, $z_k = x_k + iy_k$ and $z'_k = x'_k + iy'_k$.

Notice that for $z, z' \in \mathbb{C}^n$,

$$\langle z, z' \rangle = \langle z, z' \rangle_{\mathbb{R}} + i\langle z, iz' \rangle_{\mathbb{R}}.$$

As we will show in the proof of Theorem 9, the semitameness of f/g guarantees the existence of a vector field v on X such that:

(i) For all $z \in X$, $\langle v(z), \operatorname{grad} \log F(z) \rangle = +1$.
(ii) For all $z \in X \setminus M(F)$, $\langle v(z), z \rangle > 0$.
(iii) For all $z \in U$, $\operatorname{Re}\langle v(z), z \rangle > 0$.

So that conditions (1) and (2) are satisfied. We now introduce an additional hypothesis which will ensure that (3) is also satisfied, *i.e.* that v is such that :

(iv) For all $z \in X \cap N_\eta \setminus I$ one has $v(z) \in T_z \partial N_{\eta'}$, where ${\eta'}^2 = |f(z)|^2 + |g(z)|^2$.

As shown in the proof of the Theorem 9, semitameness is sufficient to define such a v in a neighbourhood of $M(F) \cap N_\eta$. Now, let $z \in N_\eta \setminus M(F)$. We set $\gamma(z) = |f(z)|^2 + |g(z)|^2$ so that

$$T_z \partial N_{\eta'} = \{ v \in \mathbb{R}^{2n} \mid \langle v, \operatorname{grad}_{\mathbb{R}} \gamma(z) \rangle_{\mathbb{R}} = 0 \}.$$

Then a vector $v \in \mathbb{R}^{2n}$ satisfies (i), (ii) and (iv) if and only if

$$\langle v, \operatorname{grad} \log F(z) \rangle = +1 \,, \langle v, z \rangle > 0 \ \text{ and } \ \langle v, \operatorname{grad}_{\mathbb{R}} \gamma(z) \rangle_{\mathbb{R}} = 0 \,.$$

Such a v exists if and only if $\operatorname{grad}_{\mathbb{R}} \gamma(z)$ does not belong to the $\mathbb{C}$-vector space generated by z and $\operatorname{grad} \log F(z)$, or equivalently by z and $\operatorname{grad} F(z)$. This makes natural the following definition. We set:

$$N(F) = \left\{ z \in U \setminus I \;\middle|\; \exists\, \lambda, \mu \in \mathbb{C},\ \operatorname{grad}_{\mathbb{R}} \gamma(z) = \lambda z + \mu \operatorname{grad} F(z) \right\}.$$

Definition 7. Let $n \geqslant 3$. We say that $f/g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is (IND)-tame if there exist sufficiently small constants $0 < \eta \ll \varepsilon' \ll \delta \ll \varepsilon \ll 1$ such that

$$\big(N(F) \cap N_\eta \cap X \big) \subset \big(M(F) \cap N_\eta \cap X \big).$$

When $n = 2$, we define the (IND)-tameness as an empty condition.

Notice that (IND)-tameness is a generic property in the following sense. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ without common factors. Then the set of indetermination points $I = \{ z \in \mathbb{C}^n \mid f(z) = g(z) = 0 \}$ has complex dimension $n - 2$. Moreover, $N(F) \cup M(F)$ is included in the set

$$P(F) = \{ z \in \mathbb{C}^n \mid \operatorname{rank} A(z) < 3 \},$$

where $A(z)$ is the matrix

$$\begin{pmatrix} \overline{\frac{\partial f}{\partial z_1}} f + g \overline{\frac{\partial g}{\partial z_1}} & \overline{\frac{\partial f}{\partial z_2}} f + g \overline{\frac{\partial g}{\partial z_2}} & \dots & \overline{\frac{\partial f}{\partial z_n}} f + g \overline{\frac{\partial g}{\partial z_n}} \\ z_1 & z_2 & \dots & z_n \\ \overline{\frac{\partial f}{\partial z_1} g - \frac{\partial g}{\partial z_1} f} & \overline{\frac{\partial f}{\partial z_2} g - \frac{\partial g}{\partial z_2} f} & \dots & \overline{\frac{\partial f}{\partial z_n} g - \frac{\partial g}{\partial z_n} f} \end{pmatrix}$$

Then $P(F)$ is generically a real analytic submanifold of $\mathbb{C}^n$ with real codimension $2n-4$. Then generically, the two germs of analytic submanifolds $(I, 0)$ and $(N(F) \cup M(F), 0)$ intersect only at 0. Therefore, when the constants $0 < \eta \ll \varepsilon' \ll \delta \ll \varepsilon \ll 1$ are sufficiently small, we obtain $P(F) \cap N_\eta \cap X = \emptyset$, and then, f/g is (IND)-tame.

Example 1. It may happen that f/g is (IND)-tame even if I is contained in $N(F) \cup M(F)$. For example, let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be defined by $f(x, y, z) = x^p$ and $g(x, y, z) = y^q$. Then the set of indetermination points of f/g is the z-axis, and the set $P(f/g)$ has equation $\det A(x, y, z) = 0$, *i.e.* :

$$zx^{p-1}y^{q-1}(|x|^{2p} + |y|^{2q}) = 0 \,.$$

Then $N(f/g)$ is included in the plane $\{z = 0\}$ and f/g is (IND)-tame, whereas $I \subset P(f/g)$. Hence f/g is also semitame.

Example 2. Let $f = f(x, y)$ and $g = g(x, y)$ be considered as germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}, 0)$. Then the set of indetermination points of f/g is again the z-axis, and the set $P(f/g)$ has equation

$$z \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \right) \left(|f|^2 + |g|^2 \right) = 0 \,.$$

Therefore f/g is (IND)-tame if and only if the jacobian curve $\{\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} = 0\}$ of the germ $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is included in the curve $\{fg = 0\}$.

On the other hand, it is easy to obtain examples with f/g semitame. For instance, with f, g as above, if we regard f/g as a map-germ at $0 \in \mathbb{C}^3$, then this is semitame if f/g is semitame as a germ from $(\mathbb{C}^2, 0)$ into $(\mathbb{C}, 0)$, since a sequence of bad points (z_k) for f/g would project on the plane $z = 0$ to a sequence of bad points for $f/g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Now, it is easy to check whether $f/g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is semitame by using the characterization of semitameness given in [1, Theorem 1] when $n = 2 :$ f/g is semitame if and only if the multilink $L_f \cup -L_g$ is fibered. This latter condition is easily checked by computing a resolution graph of the meromorphic function f/g: the multilink $L_f \cup -L_g$ is fibered if

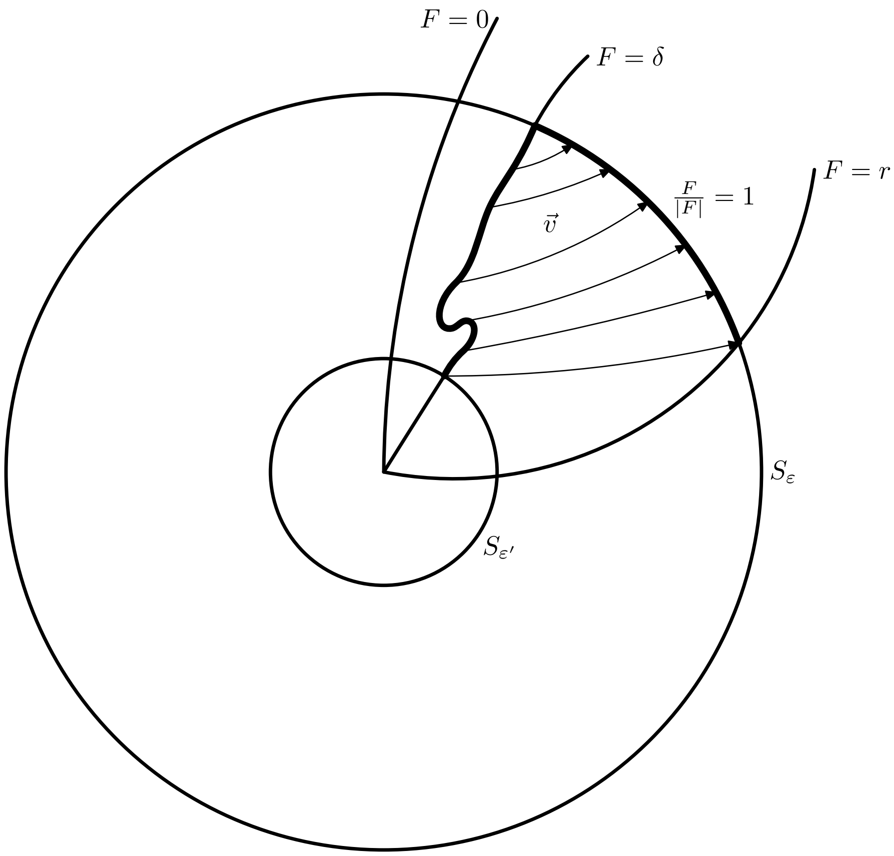

FIGURE 1. Resolution graph of $x^3 + y^2/x^2 + y^3$

and only if the multiplicities of f and g are different on each rupture component of the exceptional divisor of f/g.

Example 3. Let $f(x, y) = x^3 + y^2$ and $g(x, y) = x^2 + y^3$. Then f/g is semitame, as can be seen on the resolution graph of f/g represented on Figure 1. The number between parentheses on each vertex is the difference $m_f - m_g$ where m_f (respectively m_g) is the multiplicity of f along the corresponding component of the exceptional divisor.

But f/g, seen as a map in variables (x, y, z) is not (IND)-tame, because the germ of Jacobian curve $(J, 0)$ of (f, g) has equation $xy = 0$ and $N(f/g) = J \setminus I$.

4. The local Milnor fibrations of F

The local Milnor fibers of a meromorphic function F were defined in [3] as follows. Let us fix $c \in \mathbb{C}P^1$. There exists $\varepsilon_0 > 0$ such that for any

ε, $0 < \varepsilon \leqslant \varepsilon_0$, the restriction $F_| : \mathbb{B}_\varepsilon \setminus I \to \mathbb{C}P^1$ defines a C^∞ locally trivial fibration over a punctured neighbourhood Δ_c of the point c in $\mathbb{C}P^1$.

Definition 8. The fiber $\mathcal{M}_F^c = F^{-1}(c') \cap \mathbb{B}_\varepsilon, c' \in \Delta_c$ of this fibration is called the *c-Milnor fiber* of F. Notice that $\mathcal{M}_F^c$ is a noncompact complex $(n-1)$-dimensional manifold with boundary.

Let δ, $0 < \delta \ll \varepsilon$, be such that $\mathbb{D}_\delta(c) \subset (\Delta_c \cup \{c\})$. We call the restriction

$$\phi_c = F_| : F^{-1}(\mathbb{S}^1_\delta(c)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(c)$$

the *c-local Milnor fibration* of the meromorphic map F.

According to [3, Lemma 1], the diffeomorphism class of the noncompact $(n-1)$-complex manifold $\mathcal{M}_F^c$ does not depend on ε, and the isomorphism class of the fibration ϕ_c does not depend on ε and δ. As shown in [12], this is in fact an immediate consequence of Lê's fibration theorem in [5] applied to the pencil $\{f - tg = 0\}$.

Let c be an isolated point of B, the bifurcation set introduced in Definition 1. For our purpose it will be necessary to consider the restriction of ϕ_c to the complement in $\mathbb{B}_\varepsilon$ of a small ball $\mathbb{B}_{\varepsilon'}$, $0 \ll \varepsilon' \ll \delta$, defined as follows. We choose the punctured disc Δ_c in such a way that $\Delta_c \cap B = \emptyset$ (but usually $c \in B$). Therefore there exists ε', $0 < \varepsilon' \ll \delta \ll \varepsilon$, such that $M(F) \cap F^{-1}(\mathbb{S}^1_\delta(c)) \cap \mathbb{B}_{\varepsilon'} = \emptyset$. For such an ε', we consider the restriction of the c-local Milnor fibration

$$\check{\phi}_c = F_| : F^{-1}(\mathbb{S}^1_\delta(c)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(c)\,.$$

And we denote by $\check{\mathcal{M}}_F^c = \mathcal{M}_F^c \setminus \mathring{\mathbb{B}}_{\varepsilon'}$ the fiber of $\check{\phi}_c$ Again, the diffeomorphism class of $\check{\mathcal{M}}_F^c$ and the isomorphism class of $\check{\phi}_c$ do not depend on ε, δ and ε'.

Remark. A value $c \in \mathbb{C}P^1$ is called in [4] a typical value of the meromorphic germ F if the map $F : \mathbb{B}_\varepsilon \setminus I \to \mathbb{C}P^1$ is a locally trivial (and thus trivial) fibration over a neighbourhood of c. Let $\mathcal{B}$ be the set of values $c \in \mathbb{C}P^1$ which are not typical (called *atypical*) of F. If B is the bifurcation set defined in Definition 1 and if $c \notin B$ then, by Ehresmann's fibration theorem, one has $c \notin \mathcal{B}$. Therefore $\mathcal{B} \subset B$. In particular, if F is semitame then $\mathcal{B} \subset B \subset \{0, 1\}$.

Now let $U \subset \mathbb{C}P^1$ be the maximal open set of equisingularity in the sense of Zariski and let $B' := \mathbb{C}P^1 \setminus U$. Then obviously $B' \subset \mathcal{B}$. Moreover, when $n = 2$ one has $B' = B$ by [1, Proposition 2.11] and then $B' = \mathcal{B} = B$.

5. The results

Theorem 9. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors such that $F = f/g$ is semitame and (IND)-tame. Then for all $r > 0$ such that $B \cap \mathbb{D}_r(0) \subset \{0\}$, there exist $\varepsilon, \varepsilon'$ and δ, $0 < \varepsilon' \ll \delta \ll \varepsilon \ll 1$, such that the restricted 0-local Milnor fibration*

$$\check{\phi}_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0) \tag{5}$$

is diffeomorphic to the fibration

$$\Phi_W : \big(\mathbb{S}_\varepsilon \setminus (L_f \cup L_g)\big) \cap F^{-1}(W) \longrightarrow \mathbb{S}^1. \tag{6}$$

where $W = \mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)$.

Remember that $\check{\phi}_0$ is a restriction of F and Φ_W is a restriction of $\frac{F}{|F|}$. We are now able to draw some corollaries in the spirit of [8].

Corollary 10. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors such that $F = f/g$ is semitame and (IND)-tame. For $\delta \ll 1$ and $R \gg 1$ one has:*

- **a)** *The truncated global Milnor fiber $\check{\mathcal{M}}_F = \check{\Phi}_F^{-1}(1)$ is diffeomorphic to the union of the two restricted local Milnor fibers $\check{\mathcal{M}}^0_F = \check{\phi}_0^{-1}(\delta)$ and $\check{\mathcal{M}}^\infty_F = \check{\phi}_\infty^{-1}(R)$ glued along their boundary components $\partial_0 = \check{\phi}_0^{-1}(\delta) \cap \mathbb{S}_{\varepsilon'}$ and $\partial_\infty = \check{\phi}_\infty^{-1}(R) \cap \mathbb{S}_{\varepsilon'}$*

$$\check{\mathcal{M}}_F \simeq \check{\mathcal{M}}^0_F \cup_\partial \check{\mathcal{M}}^\infty_F .$$

- **b)** *The Euler characteristics verify:*

$$\chi\left(\check{\mathcal{M}}_F\right) = \chi\left(\check{\mathcal{M}}^0_F\right) + \chi\left(\check{\mathcal{M}}^\infty_F\right).$$

  and

$$\chi\left(\mathcal{M}_F\right) = \chi\left(\mathcal{M}^0_F\right) + \chi\left(\mathcal{M}^\infty_F\right).$$

- **c)** *The monodromies $\check{h}_0 : \check{\mathcal{M}}^0_F \to \check{\mathcal{M}}^0_F$ and $\check{h}_\infty : \check{\mathcal{M}}^\infty_F \to \check{\mathcal{M}}^\infty_F$ of the fibrations $\check{\phi}_0$ and $\check{\phi}_\infty$ are the restrictions of the monodromy $\check{h} : \check{\mathcal{M}}_F \to \check{\mathcal{M}}_F$ of the fibration $\check{\Phi}_F$.*

Proof of the Corollary. We apply Theorem 9 twice with $r = 1$. The first time as stated, the second time around ∞, or in other words, around 0 for g/f. The proof of Theorem 9 furnishes:

- a diffeomorphism Θ_0 from

$$\check{\phi}_0^{-1}(\delta) = F^{-1}(\delta) \cap \left(\mathbb{B}_\varepsilon \setminus \mathbb{B}_{\varepsilon'}\right)$$

to

$$\frac{F}{|F|}^{-1}(1) \cap \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_1(0) \setminus \mathring{\mathbb{D}}_\delta(0)),$$

such that $\Theta_0(\partial_0) = F^{-1}(1) \cap \mathbb{S}_\varepsilon$.

- and a diffeomorphism Θ_∞ from

$$\big(\check{\phi}_\infty\big)^{-1}(R) = F^{-1}(R) \cap \big(\mathbb{B}_\varepsilon \setminus \mathbb{B}_{\varepsilon'}\big)$$

to

$$\frac{F}{|F|}^{-1}(1) \cap \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_R(0) \setminus \mathring{\mathbb{D}}_1(0)),$$

such that $\Theta_\infty(\partial_\infty) = F^{-1}(1) \cap \mathbb{S}_\varepsilon$.

The intersection of the images of Θ_0 and Θ_∞ is

$$\Theta_0(\partial_0) = \Theta_\infty(\partial_\infty) = F^{-1}(1) \cap \mathbb{S}_\varepsilon\,.$$

Then $\check{\mathcal{M}}_F = \check{\Phi}_F^{-1}(1)$ is diffeomorphic to the union of $\check{\phi}_0^{-1}(\delta)$ and of $\check{\phi}_\infty^{-1}(R)$ glued along their boundary components ∂_0 and ∂_∞. This proves statement a).

The Euler characteristic verifies $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. As the intersection of the images of Θ_0 and Θ_1 is a closed oriented manifold of odd dimension, then its Euler characteristic is 0. This proves the first equation in statement b). For the second equation, notice $\mathcal{M}_F^0$ (respectively $\mathcal{M}_F^\infty$) retracts by deformation to $\check{\mathcal{M}}_F^0$ (respectively $\check{\mathcal{M}}_F^\infty$), and $\mathcal{M}_F$ retracts by deformation to $\check{\mathcal{M}}_F$, proving b).

The statement c) follows from a) and Theorem 9. □

Corollary 11. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions without common factors such that $F = f/g$ is semitame and (IND)-tame. If f, g have an isolated singularity at 0, then*

$$\chi\left(\mathcal{M}_F\right) = (-1)^{n-1}\big(\mu(f,0) + \mu(g,0) - 2\mu(f+tg,0)\big).$$

Where t is a generic value (i.e., $t \neq 0, \infty$) and μ is the Milnor number.

Proof. According to [3, Theorem 2],

$$\chi\left(\mathcal{M}_F^0\right) = (-1)^{n-1}\big(\mu(f,0) - \mu(f+tg,0)\big)$$

and

$$\chi\left(\mathcal{M}_F^\infty\right) = (-1)^{n-1}\big(\mu(g,0) - \mu(f+tg,0)\big).$$

□

Corollary 12. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs as in Corollary 11. If $n = 2$, then the manifold $\mathcal{M}_F^0$ (respectively $\mathcal{M}_F^\infty$) has the homotopy type of a bouquet of circles. If we denote by λ_0 (respectively λ_∞) the number of circles in this bouquet. Then $\mathcal{M}_F$ is a bouquet of $\lambda_0 + \lambda_\infty - 1$ circles.*

6. Preliminary lemmas

The following lemmas are easily obtained by adapting the proofs of Lemmas 4.3 and 4.4 in [6] as already performed in [7] and [1] in close situations.

Lemma 13. *Assume that the meromorphic germ $F = f/g$ is semitame at the origin. Let $p : [0, 1] \longrightarrow \mathbb{C}^n$ be a real analytic path with $p(0) = 0$ such that for all $t > 0$, $F(p(t)) \notin \{0, \infty\}$ and such that the vector* $\operatorname{grad} \log F(p(t))$ *is a complex multiple $\lambda(t)p(t)$ of $p(t)$. Then the argument of the complex number $\lambda(t)$ tends to 0 or π as $t \to 0$.*

Proof. Adapting [6, Lemma 4.4]. See also [7, Lemma 3] and [1, Lemma 2.7]. □

Lemma 14. *Let F be semitame. Then there exists $0 < \varepsilon \ll 1$ such that for all $z \in B_\varepsilon \setminus (F^{-1}(0) \cup F^{-1}(\infty))$ the two vectors z and* $\operatorname{grad} \log F(z)$ *are either linearly independent over $\mathbb{C}$ or* $\operatorname{grad} \log F(z) = \lambda z$ *with* $|\arg(\lambda)| \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$.

Proof. Using Lemma 13. See [6, Lemma 4.3] and [7, Lemma 4]. □

Lemma 15. *Let D', D'' be two 2-discs centered at 0 with $D' \subset D''$ and $D' \neq D''$. For $0 < \varepsilon \ll 1$, if $z \in \mathbb{S}_\varepsilon \setminus (F^{-1}(0) \cup F^{-1}(\infty))$ is such that* $\operatorname{grad} \log F(z) = \lambda z$, *($\lambda \in \mathbb{C}$) then*

$$F(z) \in D' \qquad \text{or} \qquad F(z) \notin D''.$$

Moreover in the first case $\arg(+\lambda) \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$ *and in the second case* $\arg(-\lambda) \in]-\frac{\pi}{4}, +\frac{\pi}{4}[$.

Proof. Using Lemma 13 and Lemma 14. See [7, Lemma 8]. □

7. Proof of the theorem

First step: definition of the constants.

(1) Let $0 < r < \infty$ be such that $B \cap \mathbb{D}_r(0) = \{0\}$, where B is the bifurcation set of the semitame[1] meromorphic function f/g.
(2) As $\mathbb{S}^1_r(0)$ is compact and $\mathbb{S}^1_r(0) \cap B = \emptyset$, one can choose $0 < \varepsilon \ll 1$ such that:

[1] I added this hypothesis

a) $\mathbb{B}_\varepsilon$ is a Milnor ball for $F^{-1}(0)$, for $F^{-1}(\infty)$, for the indetermination set I and for all $F^{-1}(z), z \in \mathbb{S}^1_r(0)$;
b) ε satisfies the conclusion of Lemma 15 for $D' = \mathbb{D}_{r/4}(0)$ and $D'' = \mathbb{D}_{r+1}(0)$.

(3) Let us choose δ, $0 < \delta \ll \varepsilon$, such that:

$$\phi_0 = F_| : F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(0)$$

is the 0-Milnor fibration of the meromorphic map F.

(4) Last, let us choose $\varepsilon'_0, 0 < \varepsilon'_0 \ll \delta$ such that

$$M(F) \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)) \cap \mathbb{B}_{\varepsilon'_0} = \emptyset,$$

and let us set $\varepsilon' = \varepsilon'_0/2$. That such an ε'_0 exists follows from the hypothesis of f/g being semitame.[2] In particular, one obtains the restricted 0-local Milnor fibration

$$\check{\phi}_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0)\,.$$

Let $\psi : U \to \mathbb{R}^2$ be defined by $\psi(z) = (\log|F(z)|, ||z||^2)$. Notice that Conditions 2.b) and 4) imply that

$$\big([\log\delta, \log r] \times [0, {\varepsilon'_0}^2] \cup [\log(r/2), \log r] \times [0, \varepsilon^2]\big) \cap \psi(M(F)) = \emptyset\,.$$

Second step : construction of a vector field.

Let us consider the set

$$X = F^{-1}\Big(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)\Big) \cap \big(\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}\big).$$

Let us fix $\rho \in]r/2, r[$ and $\varepsilon'' \in]\varepsilon', \varepsilon_0[$ and let us consider the two real numbers $0 < b_1 < b_2$ defined by:

$$b_1 = \frac{\varepsilon''^2 - \varepsilon'^2}{2(\log\rho - \log\delta)} \qquad \text{and} \qquad b_2 = \frac{\varepsilon_0^2 - \varepsilon''^2}{2(\log r - \log\rho)}\,.$$

Let us fix $\xi, 0 < \xi \ll \varepsilon''$ and an increasing C^∞ map $b : [0,\infty[\to [0,\infty[$ such that $\forall x \leqslant \varepsilon'' - \xi$, $b(x) = b_1$, and $\forall x \geqslant \varepsilon'' + \xi$, $b(x) = b_2$.

For $\eta > 0$, we consider the neighbourhood of I defined by:

$$N_\eta = \big\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 \leqslant \eta^2\big\},$$

and its boundary,

$$\partial N_\eta = \big\{z \in \mathbb{B}_\varepsilon \mid |f(z)|^2 + |g(z)|^2 = \eta^2\big\}.$$

Let us fix η_0, $0 < \eta \ll \varepsilon'$, such that η_0 satisfies the (IND)-tameness condition :

$$\big(N(F) \cap N_{\eta_0} \cap X\big) \subset \big(M(F) \cap N_{\eta_0} \cap X\big).$$

Then each $\eta \in]0, \eta_0[$ also satisfies the (IND)-tameness condition :

[2] I added this line

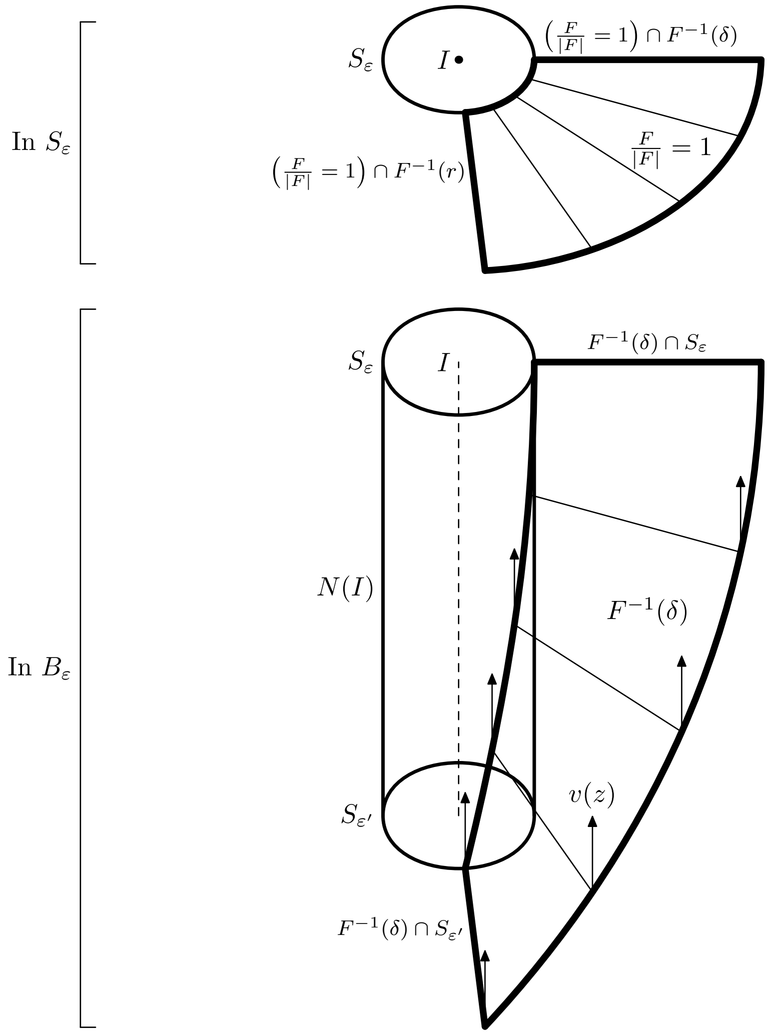

FIGURE 2. Vector field

$$\big(N(F)\cap N_\eta \cap X\big) \subset \big(M(F)\cap N_\eta\cap X\big),$$

Let us fix $\eta \in]0,\eta_0[$.

Lemma 16. *There exists an open neighbourhood Ω of the set $M(F)$ in X, two real numbers α and β, $0<\alpha<\beta$ and a differentiable vector field v on X such that:*

(i) *For all $z\in X, \langle v(z), \operatorname{grad}\log F(z)\rangle = +1$;*
(ii) *For all $z \in X\setminus M(F), \langle v(z), z\rangle = b(|z|)$;*
(iii) *For all $z\in\Omega, Re\langle v(z), z\rangle \in [\alpha,\beta]$;*
(iv) *For all $z\in X\cap N_\eta$, $v(z)\in T_z\partial N_{\eta'}$ where ${\eta'}^2 = |f(z)|^2+|g(z)|^2$.*

Proof. Let $\mu > 0$ such that $\eta + \mu \in]0, \eta_0[$. We then again have the (IND)-tameness condition :

$$\big(N(F) \cap N_{\eta+\mu} \cap X\big) \subset \big(M(F) \cap N_{\eta+\mu} \cap X\big).$$

Let us denote by V the interior of $N_{\eta+\mu}$ in X, *i.e.*,

$$V = \{z \in X \mid 0 \leqslant |f(z)|^2 + |g(z)|^2 < (\eta+\mu)^2\},$$

and let us consider the four following open sets of X (the neighbourhood Ω of $M(F)$ will be defined later) :

$$U_1 = X \setminus (N_\eta \cup M(F)), \qquad U_2 = \Omega \setminus N_\eta,$$
$$U_3 = V \cap \Omega, \qquad U_4 = V \setminus M(F)\,.$$

One has : $X = U_1 \cup U_2 \cup U_3 \cup U_4$. The vector field v will be obtained by constructing a vector field v_i on each U_i and by defining globally v by a partition of unity.

At first, let us define v on $X \setminus N_\eta = U_1 \cup U_2$. For a point $z \in U_1$, we define v_1 by using the classical construction of Milnor: for such a point the vectors z and $\operatorname{grad} \log F(z)$ are linearly independent over $\mathbb{C}$. Thus there exists $v_1(z)$ verifying (i) and (ii).

For each $z \in X$, let us consider the vector

$$u(z) = \frac{\operatorname{grad} \log F(z)}{||\operatorname{grad} \log F(z)||^2}\,.$$

Let $z \in M(F) \cap X$. There exists $\lambda \in \mathbb{C}$ such that $\operatorname{grad} \log F(z) = \lambda z$. Then $\langle u(z), \operatorname{grad} \log F(z)\rangle = +1$ and

$$\operatorname{Re}\langle u(z), z\rangle = \operatorname{Re}\left(\frac{\lambda}{|\lambda|^2}\right)\,.$$

Notice that $M(F) \cap \mathbb{B}_\varepsilon = \{z \in \mathbb{B}_\varepsilon \mid \exists \lambda \in \mathbb{C}, \operatorname{grad} F(z) = \lambda z\}$ is compact. Then $M(F) \cap X$ is a compact set, and there exist c_1, c_2, $0 < c_1 < c_2$ such that for all $z \in M(F) \cap X$ one has $c_1 < |\lambda| < c_2$ where λ is the complex number such that $\operatorname{grad} F(z) = \lambda z$. Moreover, Condition 2.b) above implies that $\arg(+\lambda) \in] -\frac{\pi}{4}, +\frac{\pi}{4}[$. Then there exists $c_1', c_2' > 0$ such that for all $z \in M(F) \cap X$, $c_1' < \operatorname{Re}\langle u(z), z\rangle < c_2'$.

Let us choose ν such that $0 < \nu \ll c_1'$ and let us set $\alpha = c_1' - \nu$ and $\beta = c_2' + \nu$. There exists an open neighbourhood Ω of $M(F)$ in X such that for all $z \in \Omega$, $\alpha < \operatorname{Re}\langle u(z), z\rangle < \beta$. Then for each $z \in U_2 = \Omega \setminus N_\eta$, we set $v_2(z) = u(z)$.

We now define v on $V = U_3 \cup U_4$, *i.e.* near the indetermination set I. A picture of the local situation near I is represented on Figure 3.

For a point z in V, we set

$$\eta' = \sqrt{|f(z)|^2 + |g(z)|^2}\,.$$

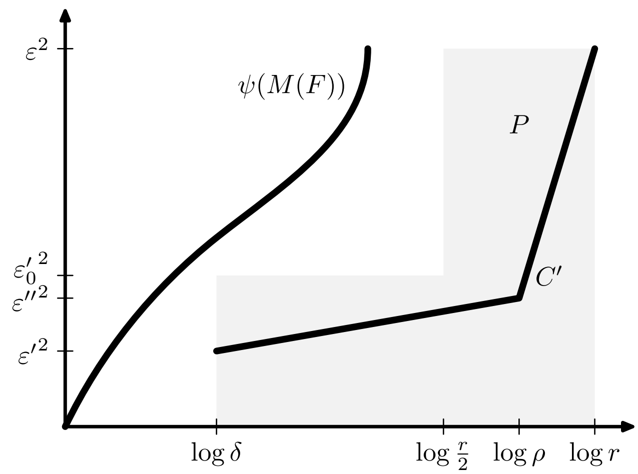

FIGURE 3. Vector field and indetermination points

Let $T = T(z)$ be the space tangent to $\partial N_{\eta'}$ at z. We will construct a vector field v on V satisfying the three conditions (i), (ii) and (iii) and such that $v(z) \in T$.

If $z \in U_3 = V \cap \Omega$, let us again consider the vector $u(z)$. If $u(z) \in T$, then we set $v_3(z) = u(z)$. If $u(z) \notin T$, let

$$Q = \big(u(z)\big)^{\perp_{\mathbb{R}}},$$

be the real orthogonal complement of the line spanned by the vector $u(z)$.

Since $\dim_{\mathbb{R}} Q = 2n-1$ and $\dim_{\mathbb{R}} T = 2n-1$, the real vector space $Q \cap T$ has dimension at least $n-2$. Let $\pi : \mathbb{R}^{2n} \to Q \cap T$ be the orthogonal projection on $Q \cap T$ in the direction of the vector $iu(z)$. We set

$$v_3(z) = \pi(u(z)).$$

Obviously $v_3(z) \in T$ and an easy computation shows that v_3 verifies conditions (i) and (iii).

Last, let us consider $U_4 = V \setminus M(F)$. Let $z \in U_4$. We set $\gamma(z) = |f(z)|^2 + |g(z)|^2$. There exists a vector $v_4(z)$ verifying (i), (ii) and $v_4(z) \in T$ if and only if the vector

$$w(z) = \operatorname{grad}_{\mathbb{R}} \gamma(z)$$

does not belong to the complex vector space H generated by the two vectors $w_1(z) = z$ and $w_2(z) = \operatorname{grad} F(z)$. This is equivalent to saying $z \notin N(F)$, which is true because F is semitame and (IND)-tame.

Now, we define globally the vector field on X by a partition of unity. $\square$

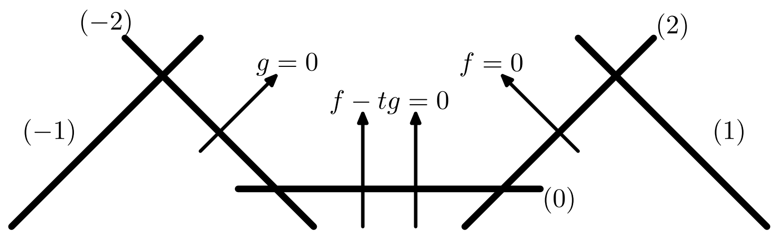

FIGURE 4. Avoidance of $M(F)$

Third step : integration of the vector field v.

We integrate the vector field v and we denote by $C = \{z = p(t)\}$ an integral curve.

Condition (i) implies that the argument of $F(p(t))$ is constant and that $|F(p(t))|$ is strictly increasing along C. Conditions (ii) and (iii) implies that $\|p(t)\|$ is strictly increasing along C.

Lemma 17. *If C pass through a point $z_0 \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{S}_{\varepsilon'})$, then C reaches $\mathbb{S}_\varepsilon$ at a point z_1 such that $|F(z_1)| = r$.*

Proof. Let C' be the arc of curve in $\mathbb{R}^2$ parametrized by $t \in [0, \log(r/\delta)]$ as follows :

- $x(t) = t + \log \delta$
- $\forall t \in [0, \log(\rho/\delta)], y(t) = 2b_1 t + \varepsilon'^2$
- $\forall t \in [\log(\rho/\delta), \log(r/\delta)], y(t) = 2b_2 t + \varepsilon''^2$

The arc C' is the union of the two segments joining the three points $(\log \delta, {\varepsilon'}^2)$, $(\log \rho, {\varepsilon''}^2)$ and $(\log r, \varepsilon^2)$. Then C' is included in the zone $P = [\log \delta, \log r] \times [0, {\varepsilon'_0}^2] \cup [\log(r/2), \log r] \times [0, \varepsilon^2]$ and $C' \cap \psi(U) = \emptyset$. (see Figure 4).

Now, let C be an integral curve of v passing through $z_0 \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{S}_{\varepsilon'}$. Then a computation analogous to that of [6] page 53, shows that C' is nothing but the image of C by ψ. Therefore, the integral curve C passing through z_0 goes transversally to the spheres centered at 0 until it reaches $\mathbb{S}_\varepsilon$ at a point belonging to $F^{-1}(\mathbb{S}^1_r(0))$. $\square$

Then, the diffeomorphism

$$\Theta_0 : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)),$$

which sends $z \in F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'})$ on the intersection $\Theta_0(z)$ of the integral curve of v passing through z with the sphere $S_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0))$, is a diffeomorphism from the fibration:

$$F : F^{-1}(\mathbb{S}^1_\delta(0)) \cap (\mathbb{B}_\varepsilon \setminus \mathring{\mathbb{B}}_{\varepsilon'}) \longrightarrow \mathbb{S}^1_\delta(0)$$

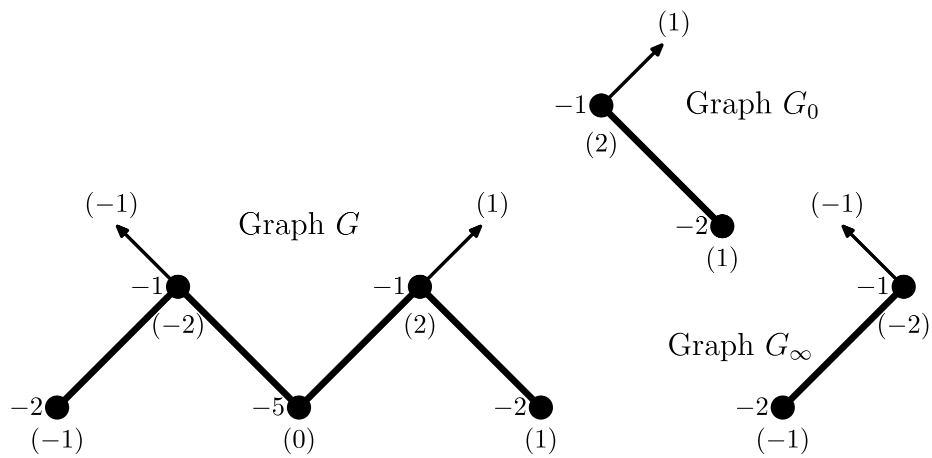

FIGURE 5. Dual resolution graph of $x^2 + y^3/x^3 + y^2$

to the fibration:

$$\Phi = \frac{F}{|F|} : \mathbb{S}_\varepsilon \cap F^{-1}(\mathbb{D}_r(0) \setminus \mathring{\mathbb{D}}_\delta(0)) \longrightarrow \mathbb{S}^1. \tag{7}$$

This completes the proof of the theorem.

8. An example

Let $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be the two holomorphic germs defined by:

$$f(x,y) = x^2 + y^3, \quad g(x,y) = x^2 + y^3.$$

Let $\pi : X \longrightarrow U$ be the resolution of the meromorphic function $F = f/g$ whose divisor is represented on Figure 1.

On Figure 5 we draw its dual graph G. The numbers between parentheses are the multiplicities of F along the corresponding component of the total transform of fg by π, *i.e.*, the $(m_i^f - m_i^g)$ where m_i^f and m_i^g are the multiplicities of $f \circ \pi$ and $g \circ \pi$. The numbers without parentheses are the Euler classes. The arrows are for the strict transforms of f and g. The strict transform of a generic fibre of F passes through the dicritical component of the exceptional locus, *i.e.* the one carrying multiplicity 0.

The meromorphic function f/g is semitame, and (IND)-tame ($n = 2$). One therefore has three different fibrations: the global Milnor fibration of f/g,

$$\Phi_F = \frac{f/g}{|f/g|} : \mathbb{S}_\varepsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1,$$

and the two local Milnor fibrations $\phi_0 = F_{|} : F^{-1}(\mathbb{S}^1_\delta(0)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(0)$ and $\phi_\infty = F_{|} : F^{-1}(\mathbb{S}^1_\delta(\infty)) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}^1_\delta(\infty)$

Using the fibration theorem for plumbed multilinks (see e.g. [11, 2.11]), one observes three different fibred multilinks in plumbing manifolds on this configuration :

(1) The link $L_f - L_g$ in the sphere $\mathbb{S}^3$.
(2) The link L_f in the plumbed manifold V_0 whose graph G_0 is the subgraph of G determined by the divisor $E_2 \cup E_3$.
(3) The link L_g in the plumbed manifold V_∞ whose graph G_∞ is the subgraph of G determined by the divisor $E_3 \cup E_4$.

As already mentioned, the map Φ_F is a fibration of the link $L_f - L_g$ in the sphere $\mathbb{S}^3$. The two local fibrations ϕ_0 and ϕ_∞ are the restrictions to the complementary of the indetermination set I of f/g of fibrations $\hat{\phi}_0$ and $\hat{\phi}_\infty$ of the links $L_f \subset V_0$ and $L_g \subset V_\infty$.

The fibres $\hat{\mathcal{M}}^0_F$ and $\hat{\mathcal{M}}^\infty_F$ of $\hat{\phi}_0$ and $\hat{\phi}_\infty$ can be computed by the Hurwitz formula from the graphs G_0 and G_∞. One obtains for both a sphere with one hole. The fibre $\mathcal{M}^0_F$ (resp. $\mathcal{M}^\infty_F$) is then obtained by removing a neighbourhood of the intersection of $\hat{\mathcal{M}}^0_F$ with $\pi^{-1}(0)$. One then obtains a sphere with three holes in both cases. Now the fiber of Φ_F is homeomorphic to the surface obtained by gluing together $\mathcal{M}^0_F$ and $\mathcal{M}^\infty_F$ along the two boundary components just created.

At last, let us recall that the isomorphism classes of the fibrations $\Phi_F, \hat{\phi}_0$ and $\hat{\phi}_\infty$ are completely described by the Nielsen invariants of their monodromies $h : \mathcal{M}_F \to \mathcal{M}_F$, $\hat{h}_0 : \hat{\mathcal{M}}^0_F \to \hat{\mathcal{M}}^0_F$ and $\hat{h}_\infty : \hat{\mathcal{M}}^\infty_F \to \hat{\mathcal{M}}^\infty_F$ (see e.g. [9]). The set of Nielsen invariants is equivalent to the data of the graph G, G_0 and G_∞ respectively, weighted by the genus and the multiplicities.

In particular, it should be mentioned that the Dehn twist performed by h on the two gluing curves $\partial_0 \subset \mathcal{M}_F$ is positive (it equals $+\frac{5}{2}$ according, for instance, to the formula (3) of [9, Lemma 4.4]), whereas all the Dehn twists performed by the monodromy of the Milnor fibration associated with a holomorphic germ are negative. This is a general phenomena in the case $n = 2$: each dicritical component of the exceptional divisor in the resolution of f/g gives rise to a positive Dehn twist along each of the corresponding separating curves on the fiber, see [2, Chapter 5].

The previous arguments show that in this example, the monodromy of the global Milnor fibration of the meromorphic germ f/g can not possibly be the monodromy of a holomorphic germ. Then, a natural question is to ask whether something similar happens in higher dimensions. That is, is there some property that distinguishes the monodromy of the global Milnor fibration of a meromorphic function from those of holomorphic germs?

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Arnaud Bodin (Arnaud.Bodin@math.univ-lille1.fr)
Laboratoire Paul Painlevé,
Université de Lille 1,
59655 Villeneuve d'Ascq, France.

Anne Pichon (pichon@iml.univ-mrs.fr)
Institut de Mathématiques de Luminy,
UMR 6206 CNRS, Campus de Luminy Case 907,
13288 Marseille Cedex 9, France.

José Seade (jseade@matcuer.unam.mx),
Instituto de Matemáticas, Unidad Cuernavaca,
Universidad Nacional Autónoma de México,
Av. Universidad s/n, Lomas de Chamilpa,
62210 Cuernavaca, Morelos, A. P. 273-3, México.