Irregular fibers of complex polynomials in two variables

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Introduction

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial. The *bifurcation set* $\mathcal{B}$ for f is the minimal set of points of $\mathbb{C}$ such that $f : \mathbb{C}^n \setminus f^{-1}(\mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a locally trivial fibration. For $c \in \mathbb{C}$, we denote the fiber $f^{-1}(c)$ by F_c. The fiber F_c is *irregular* if c is in $\mathcal{B}$. If $s \notin \mathcal{B}$, then F_s is a *generic fiber* and is denoted by F_{gen}. The *tube* T_c for the value c is a neighborhood $f^{-1}(D^2_\varepsilon(c))$ of the fiber F_c, where $D^2_\varepsilon(c)$ stands for a 2-disk in $\mathbb{C}$, centered at c, of radius $\varepsilon \ll 1$. We assume that affine critical singularities are isolated. The value c is *regular at infinity* if there exists a compact set K of $\mathbb{C}^n$ such that the restriction of f, $f : T_c \setminus K \longrightarrow D^2_\varepsilon(c)$ is a locally trivial fibration.
Set $n = 2$. Let $j_c : H_1(F_c) \longrightarrow H_1(T_c)$ be the morphism induced by the inclusion of F_c in T_c. The first part of this work is the study of this morphism. Let G_c the dual graph of $F_c = f^{-1}(c)$, and $\bar{G}_c$ the dual graph of a compactification of the fiber F_c obtained by a resolution at infinity of f. The value c is *acyclic* if the dual graph G_c and some dual graphs $G_{c,P}$ obtained by compactification have the same number of cycles (see the full definition later). This is a combinatoric condition, for example if the fiber F_c is connected then c is acyclic if and only if $H_1(G_c)$ is isomorphic to $H_1(\bar{G}_c)$. Finally we define $j_\infty : H_1(F_c \setminus K) \longrightarrow H_1(T_c \setminus K)$ induced by inclusion .

Theorem.

(A) j_c is injective if and only if F_c is connected and c is acyclic.

(B) j_c is surjective if and only if j_∞ is surjective and c is acyclic.

(C) j_c is an isomorphism if and only if c is a regular value at infinity.

E. Artal-Bartolo, Pi. Cassou-Noguès and A. Dimca have proved the part (C) in [ACD] for polynomials with a connected fiber F_c. In fact we have a stronger result for the part (A) because the rank of the kernel of j_c is: $\operatorname{rk}\operatorname{Ker} j_c = n(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c)$ where $n(F_c)$ is the number of connected components of F_c.

We apply these results to the study of neighborhoods of irregular fibers. Set $n \geqslant 2$. Let F_c° be the *smooth part* of F_c: F_c° is obtained by intersecting F_c with a large $2n$-ball and cutting out a small neighborhood of the (isolated) singularities. Then F_c° can be embedded in F_{gen}. We study the following commutative diagram that links the three elements F_c°, F_{gen}, and T_c:

$$\begin{array}{ccc} H_q(F_c^\circ) & \xrightarrow{j_c^\circ} & H_q(T_c) \\ {\scriptstyle \ell_c}\downarrow & \nearrow{\scriptstyle k_c} & \\ H_q(F_{gen}) & & \end{array}$$

where ℓ_c is the morphism induced in integral homology by the embedding; j_c° and k_c are induced by inclusions. The morphism k_c is well-known and $V_q(c) = \operatorname{Ker} k_c$ are *vanishing cycles* for the value c. Let h_c be the monodromy induced on $H_q(F_{gen})$ by a small circle around the value c. Then we prove that the image of ℓ_c are invariant cycles by h_c:

$$\operatorname{Ker}(h_c - \mathrm{id}) = \ell_c\big(H_q(F_c^\circ)\big).$$

This formula for the case $n = 2$ has been obtained by F. Michel and C. Weber in [MW]. Finally we give a description of vanishing cycles with respect to eigenvalues of h_c for homology with complex coefficients. For $\lambda \neq 1$ and p a large integer the characteristic space $E_\lambda = \operatorname{Ker}(h_c - \lambda\,\mathrm{id})^p$ is composed of vanishing cycles for the value c. For $\lambda = 1$ the situation is different. If $K_q(c) = V_q(c) \cap \operatorname{Ker}(h_c - \mathrm{id})$ are invariant and vanishing cycles we have

$$K_q(c) = \ell_c\big(\operatorname{Ker} j_c^\circ\big).$$

And for $n = 2$ we get the formula

$$\operatorname{rk} K_1(c) = r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c).$$

In the view of [DN], vanishing cycles are important: the monodromy $h_\infty : H_1(F_{gen}) \longrightarrow H_1(F_{gen})$ induces by a large circle around the set $\mathcal{B}$ and Broughton's decomposition $H_1(F_{gen}) = \bigoplus_{c\in\mathcal{B}} V_1(c)$ determine the monodromy representation $\pi_1(\mathbb{C}\backslash\mathcal{B}) \longrightarrow \mathrm{Aut} H_1(F_{gen})$. The former formula for $\operatorname{rk} K_1(c)$ enables us to describe where the vanishing cycles are with respect to a decomposition of the homology of the generic fiber given by the resolution of singularities.

1 Irregular fibers and tubes

1.1 Bifurcation set

We can describe the bifurcation set $\mathcal{B}$ as follows: let $\mathrm{Sing} = \big\{z \in \mathbb{C}^n \mid \mathrm{grad}_f(z) = 0\big\}$ be the set of *affine critical points* and let $\mathcal{B}_{\mathit{aff}} = f(\mathrm{Sing})$ be the set of *affine critical values*. The set $\mathcal{B}_{\mathit{aff}}$ is a subset of $\mathcal{B}$. The value $c \in \mathbb{C}$ is *regular at infinity* if there exists a disk D centered at c and a compact set K of $\mathbb{C}^n$ with a locally trivial fibration $f : f^{-1}(D) \setminus K \longrightarrow D$. The non-regular values at infinity are the *critical values at infinity* and are collected in $\mathcal{B}_\infty$. The finite set $\mathcal{B}$ of critical values is now:

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

In this article we always assume that **affine singularities are isolated**, that is to say that Sing is an isolated set in $\mathbb{C}^n$. For $n = 2$ this hypothesis implies that the generic fiber is a connected set.

1.2 Preleminaries

In this paragraph $n = 2$. The inclusion of F_c in T_c induces a morphism $j_c : H_1(F_c) \longrightarrow H_1(T_c)$. We firstly recall notations and results from [ACD].

Let denote $F_{aff} = F_c \cap B^4_R$ ($R \gg 1$) and $F_\infty = \overline{F_c \setminus F_{aff}}$, thus $F_{aff} \cap F_\infty = K_c = f^{-1}(c) \cap S^3_R$ is the *link at infinity* for the value c. Similarly $T_{aff} = T_c \cap B^4_R$ and $T_\infty = \overline{T_c \setminus T_{aff}}$. We denote $j_\infty : H_1(F_\infty) \longrightarrow H_1(T_\infty)$ the morphism induced by inclusion. The morphism $j_{aff} : H_1(F_{aff}) \longrightarrow H_1(T_{aff})$ is an isomorphism. $H_1(F_{aff} \cap F_\infty)$ and $H_1(T_{aff} \cap T_\infty)$ are isomorphic.

Mayer-Vietoris exact sequences for the decompositions $F_c = F_{aff} \cup F_\infty$ and $T_c = T_{aff} \cup T_\infty$ give the commutative diagram $(\mathcal{D})$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(F_\infty \cap F_{aff}) & \xrightarrow{g} & H_1(F_\infty) \oplus H_1(F_{aff}) & \xrightarrow{h} & H_1(F_c) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow j_\infty \oplus j_{aff} & & \downarrow j_c & & \\
0 & \longrightarrow & H_1(T_\infty \cap T_{aff}) & \xrightarrow{g'} & H_1(T_\infty) \oplus H_1(T_{aff}) & \xrightarrow{h'} & H_1(T_c) & \longrightarrow & H_0(T_\infty \cap T_{aff}).
\end{array}$$

The 0 at the upper-right corner is provided by the injectivity of $H_0(F_\infty \cap F_{aff}) \longrightarrow H_0(F_\infty)$ (F_c need not to be a connected set) hence $H_0(F_\infty \cap F_{aff}) \longrightarrow H_0(F_\infty) \oplus H_0(F_{aff})$ is injective.

1.3 Resolution of singularities

To compactify the situation, for $n = 2$, we need resolution of singularities at infinity [LW]:

$$\begin{array}{ccccc}
\mathbb{C}^2 & \longrightarrow & \mathbb{C}P^2 & \xleftarrow{\pi_w} & \Sigma_w \\
f \downarrow & & \tilde{f} \downarrow & \swarrow \phi_w & \\
\mathbb{C} & \longrightarrow & \mathbb{C}P^1 & &
\end{array}$$

$\tilde{f}$ is the map coming from the homogenization of f; π is the minimal blow-up of some points on the line at infinity L_∞ of $\mathbb{C}P^2$ in order to obtain a well-defined morphism $\phi_w : \Sigma_w \longrightarrow \mathbb{C}P^1$: this the *weak resolution*. We denote $\phi_w^{-1}(\infty)$ by D_∞, and let D_{dic} be the set of components D of $\pi_w^{-1}(L_\infty)$ that verify $\phi_w(D) = \mathbb{C}P^1$. Such a D is a *dicritical component*. The *degree* of a dicritical component D is the degree of the branched covering $\phi_w : D \longrightarrow \mathbb{C}P^1$. For the weak resolution the divisor $\phi_w^{-1}(c) \cap \pi_w^{-1}(L_\infty)$, $c \in \mathbb{C}$, is a union of bamboos (possibly empty) (a *bamboo* is a divisor whose dual graph is a linear tree). The set $\mathcal{B}_\infty$ is the set of values of ϕ_w on non-empty bamboos with the set of critical values of the restriction of ϕ_w to the dicritical components.
We can blow-up more points to obtain the *total resolution*, $\phi_t : \Sigma_t \longrightarrow \mathbb{C}P^1$, such that all fibers of ϕ_t are normal crossing divisors that intersect the dicritical components transversally; moreover we blow-up affine singularities. Then $D_\infty = \phi_t^{-1}(\infty)$ is the same as above and for $c \in \mathcal{B}$ we denote D_c the divisor $\phi_t^{-1}(c)$.
The *dual graph* $\bar{G}_c$ of D_c is obtained as follows: one vertex for each irreducible component of D_c and one edge between two vertices for one intersection of the corresponding components. A similar construction is done for D_∞, we know that $\bar{G}_\infty$ is a tree [LW]. The *multiplicity* of a component is the multiplicity of ϕ_t on this component.

1.4 Study of j_∞

See [ACD]. Let ϕ be the weak resolution map for f. Let denote by Dic_c the set of points P in the dicritical components, such that $\phi(P) = c$. To each $P \in \mathrm{Dic}_c$ is associated one, and only one, connected component T_P of T_∞; T_P is the *place at infinity* for P. We have $T_\infty = \coprod_{P\in\mathrm{Dic}_c} T_P$ and we set $F_P = T_P \cap F_\infty = T_P \cap F_c$ and $K_P = \partial F_P$, finally $n(F_P)$ denotes the number of connected components of F_P. Let $\bar{F}_P$ be the strict transform of c by ϕ, intersected with T_P. The study of j_∞ follows from the study of $j_P : H_1(F_P) \longrightarrow H_1(T_P)$. Let $\mathfrak{m}_P$ be the intersection multiplicity of $\bar{F}_P$ with the divisor $\pi_w^{-1}(L_\infty)$ at P.

Case of $P \in \bar{F}_P$. The group $H_1(T_P)$ is isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, M_P being the boundary of a small disk with transversal intersection with the dicritical component. Moreover if $F_P = \coprod_{i=1}^{n(F_P)} F_P^i$ then $j_P([F_P^i]) = j_P([K_P^i]) = \mathfrak{m}_P^i[M_P]$.

Case of P being in a bamboo. The group $H_1(T_P)$ is also isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, M_P being the boundary of a small disk, with transversal intersection with the last component of the bamboo. Then $j_P[F_P^i] = j_P[K_P^i] = \mathfrak{m}_P^i.\ell_i[M_P]$. The integer ℓ_i only depends of the position where F_P^i intersects the bamboo, moreover $\ell_i \geqslant 1$ and $\ell_i = 1$ if and only if F_P^i intersects the bamboo at the last component. For a computation of ℓ_i, refer to [ACD].

As a consequence j_P is injective if and only if $n(F_P) = 1$ and j_∞ is injective if and only if $n(F_P) = 1$ for all P in Dic_c. In fact the rank of the kernel of j_∞ is the sum of the ranks of the kernels of j_P then

$$\operatorname{rk}\ker j_\infty = \sum_{P\in\mathrm{Dic}_c} (n(F_P) - 1).$$

Finally j_∞ is surjective if and only if for all $P \in \mathrm{Dic}_c$, j_P is surjective.

1.5 Acyclicity

The value c is *acyclic* if the morphism $\psi : H_0(T_\infty \cap T_{\mathit{aff}}) \longrightarrow H_0(T_\infty) \oplus H_0(T_{\mathit{aff}})$ given by the Mayer-Vietoris exact sequence is injective.

Let give some interpretations of the acyclicity condition.

1. The injectivity of ψ can be view as follows: two branches at infinity that intersect the same place at infinity have to be in different connected components of F_c.

2. Let G_c be the *dual graph* of F_c (one vertex for an irreducible component of F_c, two vertices are joined by an edge if the corresponding irreducible components have non-empty intersection, if a component has auto-intersection it provides a loop) and let $G_{c,P}$ be the graph obtained from G_c by adding edges to vertices that correspond to the same place at infinity T_P. In other words c is acyclic if and only if there is no new cycles in $G_{c,P}$, that is to say $H_1(G_c) \cong H_1(G_{c,P})$ for all P in Dic_c.

3. Another interpretation is the following: c is acyclic if and only if the morphism h' of the diagram $(\mathcal{D})$ is surjective. This can be proved by the exact sequence:

$$H_1(T_\infty) \oplus H_1(T_{\mathit{aff}}) \xrightarrow{h'} H_1(T_c) \xrightarrow{\varphi} H_0(T_\infty \cap T_{\mathit{aff}}) \xrightarrow{\psi}$$
$$\xrightarrow{\psi} H_0(T_\infty) \oplus H_0(T_{\mathit{aff}}) \longrightarrow H_0(T_c).$$

4. Let consider the above Mayer-Vietoris exact sequence in reduced homology, the morphism $\widetilde{\psi} : \widetilde{H}_0(T_\infty \cap T_{\mathit{aff}}) \longrightarrow \widetilde{H}_0(T_\infty) \oplus \widetilde{H}_0(T_{\mathit{aff}})$ is surjective because $\widetilde{H}_0(T_c) = \{0\}$. Moreover $\widetilde{\psi}$ is injective if and only if ψ is injective. As $\widetilde{\psi}$ is surjective, $\widetilde{\psi}$ is injective if and only if $\operatorname{rk} \widetilde{H}_0(T_\infty \cap T_{\mathit{aff}}) = \operatorname{rk} \widetilde{H}_0(T_\infty) + \operatorname{rk} \widetilde{H}_0(T_{\mathit{aff}})$, that is to say c is acyclic if and only if

$$\sum_{P \in \mathrm{Dic}_c} n(F_P) \quad - \quad 1 = \#\mathrm{Dic}_c - 1 + n(F_c) - 1. \qquad (\star)$$

This implies the lemma:

Lemma 1. *j_∞ is injective $\iff$ F_c is a connected set and c is acyclic.*

Proof. If j_∞ is injective then $n(F_P) = 1$ for all P in Dic_c, then $H_0(T_\infty \cap T_{\mathit{aff}}) \cong H_0(T_\infty)$ and ψ is injective, hence c is acyclic and from equality $(\star)$, we have $n(F_c) = 1$ *i.e.* F_c is a connected set. Conversely, if c is acyclic and $n(F_c) = 1$ then equality $(\star)$ gives $n(F_P) = 1$ for all P in Dic_c, thus j_∞ is injective. □

Let us define a stronger notion of acyclicity. Let $\bar{G}_c$ be the dual graph of $\phi^{-1}(c)$. The graph $\bar{G}_c$ can be obtained from G_c by adding edges between vertices that belong to the same place at infinity for all P in Dic_c. The value c is *strongly acyclic* if $H_1(\bar{G}_c) \cong H_1(G_c)$. Strong acyclicity implies acyclicity, but the converse can be false. However if F_c is a connected set (that is to say G_c is a connected graph) then both conditions are equivalent. This is implicitly expressed in the next lemma, which is just a result involving graphs.

Lemma 2. $\operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c) = \sum_{P \in \mathrm{Dic}_c} \big(n(F_P) - 1\big) \; - \; \big(n(F_c) - 1\big).$

1.6 Surjectivity

Part (B). *j_c surjective $\iff$ j_∞ surjective and c acyclic.*

Proof. Let us suppose that j_c is surjective then a version of the five lemma applied to diagram $(\mathcal{D})$ proves that j_∞ is surjective. As j_c and j_∞ are surjective, diagram $(\mathcal{D})$ implies that $h' : H_1(T_\infty) \oplus H_1(T_{\mathit{aff}}) \longrightarrow H_1(T_c)$ is surjective, that means that c is acyclic. Conversely if j_∞ is surjective and c is acyclic then h' is surjective and diagram $(\mathcal{D})$ implies that j_c is surjective. □

1.7 Injectivity

Part (A). *j_c is injective $\Longleftrightarrow F_c$ is a connected set and c is acyclic.*

It follows from lemma 1 and from the next lemma.

Lemma 3. *j_c injective $\Longleftrightarrow j_\infty$ injective.*
Moreover the rank of the kernel is:

$$\begin{aligned}\operatorname{rk}\ker j_c = \operatorname{rk}\ker j_\infty &= \sum_{P\in \mathrm{Dic}_c} \big(n(F_P)-1\big)\\ &= n(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c).\end{aligned}$$

Proof. The first part of this lemma can be proved by a version of the five lemma. However we shall only prove the equality of the ranks of $\ker j_c$ and $\ker j_\infty$. It will imply the lemma because we already know that $\operatorname{rk}\ker j_\infty = \sum_{P\in\mathrm{Dic}_c}\big(n(F_P)-1\big)$ and from lemma 2 we then have $\operatorname{rk}\ker j_\infty = n(F_c)-1+\operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk} H_1(G_c)$.

The study of the morphism $j_c : H_1(F_c) \longrightarrow H_1(T_c)$ is equivalent to the study of the morphism $H_1(T_{\mathit{aff}}) \longrightarrow H_1(T_c)$ induced by inclusion that, by abuse, will also be denoted by j_c. To see this, it suffices to remark that F_c is obtained from $F_{\mathit{aff}} = F_c \cap B^4_R$ by gluing $F_c \cap S^3_R \times [0,+\infty[$ to its boundary $F_c \cap S^3_R$. Then the morphism $H_1(F_{\mathit{aff}}) \longrightarrow H_1(F_c)$ induced by inclusion is an isomorphism; finally $j_{\mathit{aff}} : H_1(F_{\mathit{aff}}) \longrightarrow H_1(T_{\mathit{aff}})$ is also an isomorphism. The long exact sequence for the pair (T_c, T_{aff}) is:

$$H_2(T_c) \longrightarrow H_2(T_c, T_{\mathit{aff}}) \longrightarrow H_1(T_{\mathit{aff}}) \xrightarrow{j_c} H_1(T_c)$$

but $H_2(T_c) = 0$ (see [ACD] for example) then the rank of $\ker j_c$ is the rank of $H_2(T_c, T_{\mathit{aff}})$. On the other hand, the study of $j_\infty : H_1(F_\infty) \longrightarrow H_1(T_\infty)$ is the same as the study of $H_1(\partial T_\infty) \longrightarrow H_1(T_\infty)$ induced by inclusion (and denoted by j_∞) because the morphisms $H_1(\partial F_\infty) \longrightarrow H_1(F_\infty)$ and $H_1(\partial F_\infty) \longrightarrow H_1(\partial T_\infty)$ induced by inclusions are isomorphisms. The long exact sequence for $(T_\infty, \partial T_\infty)$ is:

$$H_2(T_\infty) \longrightarrow H_2(T_\infty, \partial T_\infty) \longrightarrow H_1(\partial T_\infty) \xrightarrow{j_\infty} H_1(T_\infty).$$

As $H_2(T_\infty) = 0$ (see [ACD]), then the rank of $\ker j_\infty$ is the same as $H_2(T_\infty, \partial T_\infty)$.
Finally the groups $H_2(T_\infty, \partial T_\infty)$ and $H_2(T_c, T_{\mathit{aff}})$ are isomorphic by excision, and then the ranks of $\ker j_c$ and of $\ker j_\infty$ are equal. That completes the proof. □

1.8 Bijectivity

Part (C). *j_c is an isomorphism $\Longleftrightarrow c \notin \mathcal{B}_\infty$*

Proof. If $c \notin \mathcal{B}_\infty$, then the isomorphism $j_{\mathit{aff}} : H_1(F_{\mathit{aff}}) \longrightarrow H_1(T_{\mathit{aff}})$ implies that j_c is an isomorphism. Let suppose that c is a critical value at infinity and that j_c is injective. We have to prove that j_c is not surjective. As j_c is injective then by lemma 3, j_∞ is injective. By the part (B) it suffices to prove that j_∞ is not surjective. Let P be a point of Dic_c that provides irregularity at infinity for the value c, then $n(F_P) = 1$ because j_∞ is injective. Let us prove that the morphism j_P is not surjective. For the case of $P \in \bar{F}_P$, the intersection multiplicity $\mathfrak{m}_P$ is greater than 1, then j_P is not surjective. For the second case, in which P belongs to a bamboo, then $\mathfrak{m}_P.\ell_i > 1$ except for the situation where only one strict transform intersects the bamboo at the last component. This is exactly the situation excluded by the lemma "bamboo extremity fiber" of [MW]. Hence j_∞ is not surjective and j_c is not an isomorphism. □

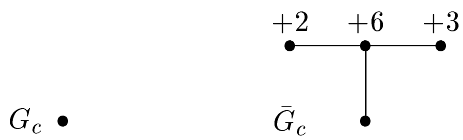
1.9 Examples

We apply the results to two classical examples.

Broughton polynomial. Let $f(x,y) = x(xy+1)$, then $\mathcal{B}_{\mathit{aff}} = \varnothing$, $\mathcal{B} = \mathcal{B}_\infty = \{0\}$. Then for $c \neq 0$, j_c is an isomorphism. The value 0 is acyclic since $H_1(G_0) \cong H_1(\bar{G}_0)$. The fiber F_0 is not connected hence j_0 is not injective. As the new component of $\bar{G}_0$ is of multiplicity 1 the corresponding morphism j_∞ is surjective, hence j_0 is surjective.

Briançon polynomial. Let $f(x,y) = yp^3 + p^2s + a_1ps + a_0s$ with $s = xy+1$, $p = x(xy+1)+1$, $a_1 = -\frac{5}{3}$, $a_0 = -\frac{1}{3}$. The bifurcation set is $\mathcal{B} = \mathcal{B}_\infty = \{0, c = -\frac{16}{9}\}$, moreover all fibers are smooth and irreducible. The value 0 is not acyclic then j_0 is neither injective nor surjective (but j_∞ is surjective).

The value c is acyclic, and F_c is connected (since irreducible) then j_c is injective. The morphism j_c is not surjective: j_∞ is not surjective because the compactification of F_c does not intersect the bamboo at the last component.



2 Situation around an irregular fiber

For $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ we study the neighborood of an irregular fiber.

2.1 Smooth part of F_c

Let fix a value $c \in \mathbb{C}$ and let B_R^{2n} be a large closed ball ($R \gg 1$). Let $B_1^{2n}, \ldots, B_p^{2n}$ be small open balls around the singular points (which are supposed to be isolated) of F_c :

$F_c \cap \mathrm{Sing}$. We denote $B_1^{2n} \cup \ldots \cup B_p^{2n}$ by $B_\cup$. Then the *smooth part* of F_c is

$$F_c^\circ = F_c \cap B_R^{2n} \setminus B_\cup.$$

It is possible to embed F_c° in the generic fiber F_{gen} (see [MW] and [NN]). We now explain the construction of this embedding by W. Neumann and P. Norbury. As F_c has transversal intersection with the balls of $B_\cup$ and with B_R^{2n}, then there exists a small disk $D_\varepsilon^2(c)$ such that for all s in this disk, F_s has transversal intersection with these balls. According to Ehresmann fibration theorem, f induces a locally trivial fibration

$$f_| : f^{-1}\big(D_\varepsilon^2(c)\big) \cap B_R^{2n} \setminus B_\cup \longrightarrow D_\varepsilon^2(c).$$

In fact, as $D_\varepsilon^2(c)$ is null homotopic, this fibration is trivial. Hence $F_c^\circ \times D_\varepsilon^2(c)$ is diffeomorphic to $f^{-1}(D_\varepsilon^2(c)) \cap B_R^{2n} \setminus B_\cup$. That provides an embedding of F_c° in F_s for all s in $D_\varepsilon^2(c)$; and for such a s with $s \neq c$, F_s is a generic fiber. The morphism induced in homology by this embedding is denoted by ℓ_c. Let j_c° be the morphism induced by the inclusion of F_c° in $T_c = f^{-1}(D_\varepsilon^2(c))$. Similarly k_c denotes the morphism induced by the inclusion of the generic fiber $F_{gen} = F_s$ (for $s \in D_\varepsilon^2(c)$, $s \neq c$) in T_c. As all morphisms are induced by natural maps we have the lemma:

Lemma 4. *The following diagram commutes:*

$$\begin{array}{ccc} H_q(F_c^\circ) & \xrightarrow{\;j_c^\circ\;} & H_q(T_c) \\ {\scriptstyle \ell_c}\big\downarrow & \nearrow {\scriptstyle k_c} & \\ H_q(F_{gen}) & & \end{array}$$

2.2 Invariant cycles by h_c

Invariant cycles by the monodromy h_c can be recovered by the following property.

Proposition 5.

$$\mathrm{Ker}\,\big(h_c - \mathrm{id}\big) = \ell_c\big(H_q(F_c^\circ)\big).$$

For $n = 2$, there is a similar formula in [MW], even for non-isolated singularities.

Proof. The proof uses a commutative diagram due to W. Neumann and P. Norbury [NN]:

$$\begin{array}{ccc} H_q(F_{gen}, F_c^\circ) & \xrightarrow[\psi]{\sim} & V_q(c) \\ {\scriptstyle \varphi}\big\uparrow & & \big\downarrow {\scriptstyle i}\ \subset \\ H_q(F_{gen}) & \xrightarrow{\;\mathrm{id} - h_c\;} & H_q(F_{gen}) \end{array}$$

The morphism i is the inclusion and ψ is an isomorphism, so $\mathrm{Ker}(h_c - \mathrm{id})$ equals $\mathrm{Ker}\,\varphi$. The long exact sequence for the pair (F_{gen}, F_c°) is:

$$\cdots \longrightarrow H_q(F_c^\circ) \xrightarrow{\ell_c} H_q(F_{gen}) \xrightarrow{\varphi} H_q(F_{gen}, F_c^\circ) \longrightarrow \cdots$$

So $\mathrm{Im}\,\ell_c = \mathrm{Ker}\,\varphi = \mathrm{Ker}(h_c - \mathrm{id})$. □

We are able to applicate this result to the calculus of the rank of $\mathrm{Ker}(h_c - \mathrm{id})$ for $n = 2$. Let denote the number of irreducible components in F_c by $r(F_c)$, and let Sing_c be $\mathrm{Sing} \cap F_c$: the affine singularities on F_c. Then $H_2(F_{gen}, F_c^\circ)$ has rank the cardinal of Sing_c which is also the rank of $\mathrm{Ker}\,\ell_c$. Moreover $\mathrm{rk}\, H_1(F_c^\circ) = r(F_c) - \chi(F_c) + \#\,\mathrm{Sing}_c$.

$$\begin{aligned}
\mathrm{rk}\,\mathrm{Ker}\,\big(h_c - \mathrm{id}\big) &= \mathrm{rk}\,\mathrm{Im}\,\ell_c \\
&= \mathrm{rk}\, H_1(F_c^\circ) - \mathrm{rk}\,\mathrm{Ker}\,\ell_c \\
&= r(F_c) - \chi(F_c) + \#\,\mathrm{Sing}_c - \#\,\mathrm{Sing}_c \\
&= r(F_c) - \chi(F_c).
\end{aligned}$$

Remark. We obtain the following fact (see [MW]): if the fiber F_c ($c \in \mathcal{B}$) is irreducible then $h_c \neq \mathrm{id}$. The proof is as follows: if $r(F_c) = 1$ and $h_c = \mathrm{id}$ then from one hand $\mathrm{rk}\,\mathrm{Ker}(h_c - \mathrm{id}) = \mathrm{rk}\, H_1(F_{gen}) = 1 - \chi(F_{gen})$ and from the other hand $\mathrm{rk}\,\mathrm{Ker}(h_c - \mathrm{id}) = 1 - \chi(F_c)$; thus $\chi(F_c) = \chi(F_{gen})$ which is absurd for c in $\mathcal{B}$ by Suzuki formula.

2.3 Vanishing cycles

Now and until the end of this paper homology is homology with complex coefficients.

Vanishing cycles for eigenvalues $\lambda \neq 1$. Let E_λ be the space $E_\lambda = \mathrm{Ker}(h_c - \lambda\,\mathrm{id})^p$ for a large integer p.

Lemma 6. *If $\lambda \neq 1$ then $E_\lambda \subset V_q(c)$.*

Proof. If $\sigma \in H_q(F_{gen})$ then $h_c(\sigma) - \sigma \in V_q(c)$. This is just the fact that the cycle $h_c(\sigma) - \sigma$ corresponds to the boundary of a "tube" defined by the action of the geometrical monodromy. We remark this fact can be generalized for $j \geqslant 1$ to

$$h_c^j(\sigma) - \sigma \in V_q(c).$$

Let p be an integer that defines E_λ, then for $\sigma \in E_\lambda$:

$$\begin{aligned}
0 = (h_c - \lambda\,\mathrm{id})^p(\sigma) &= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} h_c^j(\sigma) \\
&= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \big(h_c^j(\sigma) - \sigma\big) + \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \sigma \\
&= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \big(h_c^j(\sigma) - \sigma\big) + (1-\lambda)^p \sigma.
\end{aligned}$$

Each $h_c^j(\sigma) - \sigma$ is in $V_q(c)$, and a sum of such elements is also in $V_q(c)$, then $(1-\lambda)^p\sigma \in V_q(c)$. As $\lambda \neq 1$, then $\sigma \in V_q(c)$. ☐

Vanishing cycles for the eigenvalue $\lambda = 1$. We study what happens for cycles associated to the eigenvalue 1. Let recall that vanishing cycles $V_q(c) = \operatorname{Ker} k_c$ for the value c, are cycles that "disappear" when the generic fiber tends to the fiber F_c. Hence cycles that will not vanish are cycles that already exist in F_c. From the former paragraph these cycles are associated to the eigenvalue 1.
Let $(\tau_1, \ldots, \tau_p)$ be a family of $H_q(F_{gen})$ such that the matrix of h_c in this family is:

$$\begin{pmatrix} 1 & 1 & & & (0) \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & (0) & & \ddots & 1 \\ & & & & 1 \end{pmatrix}.$$

Then, the cycles $\tau_1, \ldots, \tau_{p-1}$ are vanishing cycles. It is a simple consequence of the fact that $h_c(\sigma) - \sigma \in V_q(c)$, because for $i = 1, \ldots, p-1$, we have $h(\tau_{i+1}) - \tau_{i+1} = \tau_i$, and then τ_i is a vanishing cycle. It remains the study of the cycle τ_p and the particular case of Jordan blocks (1) of size 1×1. We will start with the second part.

Vanishing and invariant cycles. Let $K_q(c)$ be invariant and vanishing cycles for the value c: $K_q(c) = \operatorname{Ker}(h_c - \mathrm{id}) \cap V_q(c)$. Let us remark that the space $K_q(c) \oplus \bigoplus_{c' \neq c} V_q(c')$ is not equal to $\operatorname{Ker}(h_c - \mathrm{id})$. But equality holds in cohomology.

Lemma 7. $K_q(c) = \ell_c(\operatorname{Ker} j_c^\circ)$.

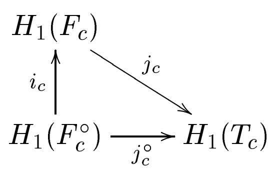
This lemma just follows from the description of invariant cycles (proposition 5) and from the diagram of lemma 4. For $n = 2$ we can calculate the dimension of $K_1(c)$.

Proposition 8. *For $n = 2$,* $\operatorname{rk} K_1(c) = r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c)$.

Proof. The proof will be clear after the following remarks:

1. $K_1(c) = \ell_c(\operatorname{Ker} j_c^\circ)$, by lemma 7.

2. $j_c^\circ = j_c \circ i_c$ with $i_c : H_1(F_c^\circ) \longrightarrow H_1(F_c)$ the morphism induced by inclusion. It is consequence of the commutative diagram:



3. $\operatorname{rk} \operatorname{Ker} j_c^\circ = \operatorname{rk} \operatorname{Ker} i_c + \operatorname{rk} \operatorname{Ker} j_c \cap \operatorname{Im} i_c$, which is general formula for the kernel of the composition of morphisms.

4. $\operatorname{Ker} j_c \cap \operatorname{Im} i_c = \operatorname{Ker} j_c$, because cycles of $H_1(F_c)$ that do not belong to $\operatorname{Im} i_c$ are cycles corresponding to $H_1(G_c)$, so they already exist in F_c and are not vanishing cycles.

5. $\operatorname{rk} \operatorname{Ker} i_c = \sum_{z \in \mathrm{Sing}_c} r(F_{c,z})$, where $F_{c,z}$ denotes the germ of the curve F_c at z.

6. $\operatorname{rk}\operatorname{Ker} j_c = \operatorname{rk}\operatorname{Ker} j_\infty = \sum_{P\in\mathrm{Dic}_c}(n(F_P)-1) = n(F_c) + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk}(G_c)$, it has been proved in lemma 3.
7. $r(F_c) + \operatorname{rk} H_1(G_c) = n(F_c) + \sum_{z\in\mathrm{Sing}_c}(r(F_{c,z})-1)$. This a general formula for the graph G_c, the number of vertices of G_c is $r(F_c)$, the number of connected components is $n(F_c)$, the number of loops is $\operatorname{rk} H_1(G_c)$ and the number of edges for a vertex that correspond to an irreducible component F_{irr} of F_c is: $\sum_{z\in F_{irr}}(r(F_{irr,z})-1)$.
8. $\operatorname{rk} K_1(c) = \operatorname{rk}\operatorname{Ker} j_c^\circ - \#\,\mathrm{Sing}_c$ because $\operatorname{Ker} i_c$ is a subspace of $\operatorname{Ker}\ell_c$ so $\operatorname{rk} K_1(c) = \operatorname{rk}\operatorname{Ker} j_c^\circ - \operatorname{rk}\operatorname{Ker}\ell_c$ and the dimension of $\operatorname{Ker}\ell_c$ is $\#\,\mathrm{Sing}_c$ (see paragraph 2.2).

We complete the proof:

$$\begin{aligned}
\operatorname{rk} K_1(c) &= \operatorname{rk}\ell_c(\operatorname{Ker} j_c^\circ) && (1)\\
&= \operatorname{rk}\operatorname{Ker} j_c^\circ - \operatorname{rk}\operatorname{Ker}\ell_c && (8)\\
&= \operatorname{rk}\operatorname{Ker} j_c\circ i_c - \#\,\mathrm{Sing}_c && (2) \text{ and } (8)\\
&= \operatorname{rk}\operatorname{Ker} i_c + \operatorname{rk}\operatorname{Ker} j_c\cap\operatorname{Im} i_c - \#\,\mathrm{Sing}_c && (3)\\
&= \operatorname{rk}\operatorname{Ker} i_c - \#\,\mathrm{Sing}_c + \operatorname{rk}\operatorname{Ker} j_c && (4)\\
&= \sum_{z\in\mathrm{Sing}_c}\big(r(F_{c,z})-1\big) + n(F_c) + \operatorname{rk} H_1(\bar{G}_c) - \operatorname{rk}(G_c) && (5) \text{ and } (6)\\
&= r(F_c) - 1 + \operatorname{rk} H_1(\bar{G}_c). && (7)
\end{aligned}$$

□

Filtration. Let ϕ be the map provided by the total resolution of f. The divisor $\phi^{-1}(c)$ is denoted by $D = \sum_i m_i D_i$ where m_i stands for the multiplicity of D_i. We associate to D_i a part of the generic fiber denoted by F_i. We briefly recall this construction (see [MW]), let $V = \phi^{-1}(D^2_\varepsilon(c))$ be a tubular neighborhood of D, we will identify the generic fiber F_{gen} with $\phi^{-1}(s)\setminus\pi^{-1}(L_\infty)$ for a generic value $s\in\partial D^2_\varepsilon(c)$, π is the blow-up. There is a natural deformation retraction $R : V \longrightarrow D$, and we set $F_i = R^{-1}(D_i)\cap F_{gen}$. The *filtration* of the homology of the generic fiber is the sequence of inclusions:

$$W_{-1}\subset W_0\subset W_1\subset W_2 = H_1(F_{gen}).$$

with

- W_{-1}: the *boundary cycles*, that is to say, if $\bar{F}_{gen}$ is the compactification of F_{gen} and $\iota_* : H_1(F_{gen}) \longrightarrow H_1(\bar{F}_{gen})$ is induced by inclusion then $W_{-1} = \operatorname{Ker}\iota_*$.
- W_0: these are *gluing cycles*: the homology group on the components of $F_i\cap F_j$ $(i\neq j)$.
- W_1: the direct sum of the $H_1(F_i)$.
- $W_2 = H_1(F_{gen})$.

The subspaces W_0 and W_1 depend on the value c.

Jordan blocks for $n = 2$. For polynomials in two variables, the size of Jordan blocks for the monodromy h_c is less or equal to 2. Let denote by σ and τ cycles of $H_1(F_{gen})$ such that $h(\sigma) = \sigma$ and $h(\tau) = \sigma + \tau$. The matrix of h_c for the family (σ, τ) is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We already know that the cycle σ vanishes.
A *large cycle* is a cycle of $W_2 = H_1(F_{gen})$ that has a non-trivial class in W_2/W_1. According to [MW] τ is large cycle; moreover large cycles associated to the eigenvalue 1 are the embedding of $H_1(\bar{G}_c)$ in $H_1(F_{gen})$. So large cycles are not vanishing cycles. The number of classes of large cycles in W_2/W_1 is $\operatorname{rk} H_1(\bar{G}_c)$, this is also the number of Jordan 2-blocks for the eigenvalue 1.

Vanishing cycles. We are now able to describe vanishing cycles. For all the spaces W_{-1}, W_0/W_{-1}, W_1/W_0 and W_2/W_1 the cycles associated to eigenvalues different from 1 are vanishing cycles.

Proposition 9. *Vanishing cycles for the eigenvalue* 1 *are dispatch as follows:*

- *for W_{-1}: $r(F_c) - 1$ cycles,*
- *for W_0: $\operatorname{rk} H_1(\bar{G}_c)$ other cycles,*
- *W_1, W_2: no other cycle.*

Proof. We have already remark that large cycles associated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are not vanishing cycles, so vanishing cycles in W_2 are in W_1. Moreover there is $\operatorname{rk} H_1(\bar{G}_c)$ Jordan 2-blocks for the eigenvalue 1 that provide $\operatorname{rk} H_1(\bar{G}_c)$ vanishing cycles (like σ) in W_0. The other vanishing cycles for the eigenvalue 1 are invariant cycles by h_c, in other words they belong to $K_1(c)$. We have $W_1 \cap K_1(c) = W_0 \cap K_1(c)$ because invariant cycles for W_1 that are not in W_0 correspond to the genus of the smooth part F_c° of F_c (this is due to the equality $\operatorname{Ker}(h_c - \mathrm{id}) = \ell_c(H_1(F_c^\circ))$). As they already appear in F_c, theses cycles are not vanishing cycles for the value c. Finally, if we have two distinct cycles σ and σ' in $W_0 \cap K_1(c)$, with the same class in W_0/W_{-1}, then $\sigma' = \sigma + \pi$, $\pi \in W_{-1}$; this implies that $\pi = \sigma' - \sigma$ is a vanishing cycle of $K_1(c)$. We can choose the $r(F_c) - 1$ remaining cycles of $K_1(c)$ in W_{-1}. □

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