JUMP OF MILNOR NUMBERS

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ABSTRACT. In this note we study a problem of A'Campo about the minimal non-zero difference between the Milnor numbers of a germ of plane curve and one of its deformation.

1. PROBLEM OF THE JUMP (A'CAMPO)

Let $f_0 : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic germ of isolated singularity. A *deformation* of f_0 is a family $(f_s)_{s\in[0,1]}$ of germs of isolated singularities such that the coefficients are analytic functions of $s \in [0, 1]$.

The *jump of the family* (f_s) is

$$\mu(f_0) - \mu(f_s), \quad 0 < s \ll 1,$$

where μ is the Milnor number at the origin. This number is well-defined because $\mu(f_0) - \mu(f_s)$ is independent of s if s is sufficiently small, moreover by the upper semi-continuity of μ this number is a non-negative integer.

The most famous result about the Milnor number and the topology of the family is Lê-Ramanujam's theorem [6]:

Theorem 1. *If $n \neq 3$ and if $\mu(f_0) = \mu(f_s)$ for all $s \in [0, 1]$ then the topological types of $f_0^{-1}(0)$ and $f_s^{-1}(0)$ are equal.*

In other words, if the jump of the family (f_s) is 0 then $f_0^{-1}(0)$ and $f_s^{-1}(0)$ have the same topological type for sufficiently small s. Another motivation is that the jump of a family is crucial in the theory of singularities of polynomial maps at infinity.

The *jump* $\lambda(f_0)$ of f_0 is the minimum of the non-zero jumps of the (f_s) over all deformations of f_0. The problem, asked by N. A'Campo, is to compute $\lambda(f_0)$. We will only deal with plane curve singularities, that is to say $n = 2$.

As a corollary of our study we prove the following:

Theorem 2. *If f_0 is an irreducible germ of plane curve and is Newton non-degenerate then*

$$\lambda(f_0) = 1.$$

A closely related question of V. Arnold [1] formulated with our definitions is to find all singularities with $\lambda(f_0) = 1$. S. Gusein-Zade [4] proved that

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there exist singularities with $\lambda(f_0) > 1$ and as a corollary of a studied of the behaviour of the Milnor number in a deformation of a desingularization he proved Theorem 2 for all irreducible plane curves.

This note is organized as follows, in paragraphs 2 to 5 we define and calculate a weak form of the jump : the non-degenerate jump. In paragraph 6 we prove Theorem 2 and in paragraph 7 we give estimations when the germ is not irreducible. Finally in paragraph 8 we state some conjectures for the jump of $x^p - y^q$, $p, q \in \mathbb{N}$ and end by questions.

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2. Kušnirenko's formula

We firstly recall some definitions (see [5]). Let $f(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j}x^iy^j$ be an analytic germ of plane curve. Let $\operatorname{supp}(f) = \{(i,j) \in \mathbb{N}^2 \mid a_{i,j} \neq 0\}$ and $\Gamma_+(f)$ be the convex closure of $\bigcup_{(i,j)}((i,j)+\mathbb{R}^2_+)$ where $(i,j) \in \operatorname{supp}(f) \setminus \{(0,0)\}$. The *Newton polygon* $\Gamma(f)$ is the union of the compact faces (called the *slopes*) of $\Gamma_+(f)$. We often identify a pair $(i,j) \in \mathbb{N}^2$ with the monomial x^iy^j. Let f be *convenient* if $\Gamma(f)$ intersects both x-axis and y-axis.

For a face γ of $\Gamma(f)$, let $f_\gamma = \sum_{(i,j)\in\gamma} a_{i,j}x^iy^j$. Then f is *(Newton) non-degenerate* if for all faces γ of $\Gamma(f)$ the system

$$\frac{\partial f_\gamma}{\partial x}(x,y) = 0 \ ; \quad \frac{\partial f_\gamma}{\partial y}(x,y) = 0$$

has no solution in $\mathbb{C}^* \times \mathbb{C}^*$.

For a Newton polygon $\Gamma(f)$, let S be the area bounded by the polygon and a (resp. b) the length of the intersection of $\Gamma(f)$ with the axes x-axis (resp. y-axis). We set

$$\nu(f) = 2S - a - b + 1.$$

For a convenient germ f the local Milnor number verifies [5] :

Theorem 3.

- $\mu(f) \geqslant \nu(f)$,
- *if f is non-degenerate then $\mu(f) = \nu(f)$.*

3. Non-degenerate jump for curve singularities

We will consider a weaker problem: Let f_0 be a plane curve singularity and we suppose that (f_s) is a *non-degenerate deformation* that is to say for all $s \in]0,1]$, f_s is Newton non-degenerate. The *non-degenerate jump* $\lambda'(f_0)$ of f_0 is the minimum of the non-zero jumps over all non-degenerate deformations of f_0. The new problem is to compute $\lambda'(f_0)$, in this note we explain how to compute it.

Obviously we have $\lambda(f_0) \leqslant \lambda'(f_0)$ but this inequality can be strict. For example let $f_0(x,y) = x^4 - y^4$, then $\lambda'(f_0) = 3$ which is obtained for the family $f'_s(x,y) = x^4 - y^4 + sx^3$. But $\lambda(f_0) \leqslant 2$, by the degenerate family $f_s(x,y) = x^4 - (y^2 + sx)^2$ of jump 2.

4. Computation of the non-degenerate jump

For a convenient f_0 there exists a finite set $\mathcal{M}$ of monomials $x^p y^q$ lying between the axes (in a large sense) and the Newton polygon $\Gamma(f_0)$ (in a strict sense).

Lemma 4. *If f_0 is non-degenerate and convenient then*

$$\lambda'(f_0) = \min_{x^p y^q \in \mathcal{M}} \big(\mu(f_0) - \mu(f_0 + s x^p y^q)\big),$$

for a sufficiently small $s \neq 0$ (the minimum is over the non-zero values).

Proof. The proof is purely combinatoric and is inspired from [2]. For any polygon T of $\mathbb{N} \times \mathbb{N}$, we define as for ν a number $\tau(T) = 2S - a - b$. Then τ is additive : let T_1, T_2 be polygons whose vertices are in $\mathbb{N} \times \mathbb{N}$, and such that $T_1 \cap T_2$ has null area then $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2)$. By this additivity we can argue on triangles only. Moreover for a polygon T that do not contain $(0, 0)$ we have $\tau(T) \geqslant 0$.

Now the jump for a non-degenerate family (f_s) corresponds to $\tau(T)$ where T is the polygon "between" $\Gamma(f_0)$ and $\Gamma(f_s)$ $(0 < s \ll 1)$. Minimizing this jump is equivalent to minimizing $\tau(T)$. It is obtained for a polygon T for which all vertices except one are in $\Gamma(f_0)$ and the last vertex is in $\Gamma(f_s)$. Then it is sufficient to add only one monomial corresponding to the latter vertex to obtain the required deformation. ☐

With this method we do not compute $\mu(f_0)$, nor $\mu(f_0 + s x^p y^q)$ but directly the difference.

For a degenerate function f we denote by $\tilde{f}$ a non-degenerate function such that f and $\tilde{f}$ have the same Newton polygon: $\Gamma(f_0) = \Gamma(\tilde{f}_0)$. The non-degenerate jump for a degenerate function f_0 can be computed with the easy next lemma:

Lemma 5. *Let f_0 be degenerate.*

- $\lambda'(f_0) = \mu(f_0) - \mu(\tilde{f}_0)$ *if* $\mu(f_0) - \mu(\tilde{f}_0) > 0$,
- *else* $\lambda'(f_0) = \lambda'(\tilde{f}_0)$.

5. An example

For a given polynomial f_0 it is very fast to see who will be the good candidates $x^p y^q$ and hence to find $\lambda'(f_0)$ after a very few calculus: we use that $\mu(f_0) - \mu(f_0 + s x^p y^q) = \tau(T)$ where T is the zone between the Newton polygon of f_0 and the one of $f_0 + s x^p y^q$.

For example let $f_0(x, y) = x^4 - y^3$. We draw its Newton polygon (see Figure 1). We easily see that the monomials $x^p y^q$ that are candidates to minimize τ for the zone between the Newton polygons are x^3 (that will give a zone with $\tau(T) = 2$) and xy^2 that will give a zone with $\tau(T) = 1$. In that case the deformation will be $f_s(x, y) = x^4 - y^3 + sxy^2$ and the jump of f_0 is 1.

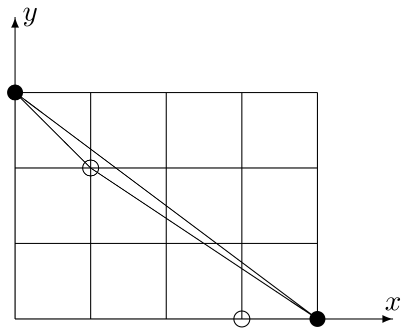

FIGURE 1. Example $f_0(x,y) = x^4 - y^3$

6. Irreducible case

In some cases we are able to give a formula for the computation of the jump. For example if $f_0(x,y) = x^p - y^q$, with $\gcd(p,q) = d$ then by Bézout theorem there exists a pair (a,b) such that $x^a y^b$ is in $\mathcal{M}$ and such that the area T corresponding to the deformation $f_s(x,y) = x^p - y^q + s x^a y^b$ is equal to $d/2$.

As an application we prove Theorem 2 cited in the introduction.

Theorem 6. *If f is irreducible and non-degenerate then $\lambda(f) = \lambda'(f) = 1$.*

Proof. We recall some facts from the book of Brieskorn-Knörrer [3, p.477]. For a germ of curve f, the number of slopes of a Newton polygon $\Gamma(f)$ is lower or equal to the number r of irreducible components.

Moreover let R be the number of lattice points that belongs to $\Gamma(f)$ minus 1. Then if f is non-degenerate we have $R = r$. The non-degenerate condition is not explicit in [3] but it is stated with an equivalent condition (a face is non-degenerate if and only if the corresponding polynomial g_i of [3, p.478] has only simple roots).

Then for an irreducible singular germ f, $\Gamma(f)$ has only one slope and f is convenient; moreover if f is non-degenerate then the extremities of $\Gamma(f)$, say x^p and y^q, verify $\gcd(p,q) = 1$. The non-degenerate jump of f is the same as for $f_0 = x^p - y^q$ and is equal to 1 by Bézout theorem. Then $\lambda'(f) = \lambda'(f_0) = 1$, as $0 < \lambda(f) \leqslant \lambda'(f) = 1$ it implies $\lambda(f) = 1$. □

7. Non irreducible case

More generally if f is convenient, non-degenerate, with one slope, let x^p, y^q be the extremities of the Newton polygon of f. Then f has the same non-degenerate jump as $f_0 = x^p - y^q$, we suppose $p \geqslant q$ and we set $d = \gcd(p,q)$. The formula for $\lambda'(f_0)$ is given by:

(1) If $1 \leqslant d < q \leqslant p$ then $\lambda'(f_0) = d$ which is reached by a family $f_s(x,y) = x^p - y^q + s x^a y^b$, a, b given by Bézout theorem.
(2) If $\gcd(p,q) = q$, i.e. $d = q$ then $\lambda'(f_0) = q - 1$ which is reached with $f_s(x,y) = x^p - y^q + s x^{p-1}$.

We will give in paragraph 8 a conjectural value for $\lambda(x^p - y^q)$.

If there are several slopes with f convenient and non-degenerate then we can estimate $\lambda'(f)$. Let $f = \prod_{i=1}^{k} f_i$ be the decomposition of f according to the slopes of $\Gamma(f)$ (notice that f_i is not necessarily irreducible). If f_i is a smooth germ then we set (by convention) $\lambda'(f_i) = 1$. In fact f_i is smooth if and only if the corresponding slope Γ_i with extremities A_i, B_i verifies $|x_{B_i} - x_{A_i}| = 1$ or $|y_{B_i} - y_{A_i}| = 1$. Then the following can be proved:

Lemma 7. *Let f be a convenient non-degenerate germ with several slopes, let $f = \prod_{i=1}^{k} f_i$ be the decomposition according to the slopes.*

1. *If all the f_i are smooth then $\lambda'(f) = 1$.*
2. *If none of the f_i is smooth then*
$$\min_{i=1..k} \lambda'(f_i) \leqslant \lambda'(f) \leqslant \max_{i=1..k} \lambda'(f_i).$$
3. *In the other cases we have*
$$\min_{i=1..k} \lambda'(f_i) \leqslant \lambda'(f) \leqslant \max_{i=1..k} \lambda'(f_i) + 1.$$

We give some examples:

1. The family $f_s(x,y) = (x+y^4)(x+y^2)(x^2+y) + sy^4$ is of non-degenerate jump 1.
2. The family $f_s(x,y) = (x^8-y^6)(x^3-y^2) + sxy^7$ gives $\lambda'(f_0) = 2$ with $\lambda'(x^8-y^6) = 2$ and $\lambda'(x^3-y^2) = 1$.
3. The family $f_s(x,y) = (x^8-y^6)(x^3-y^2)(x^4-y^4) + sx^5y^7$ verifies $\lambda'(f_0) = 2$ while $\lambda'(x^8-y^6) = 2$ and $\lambda'(x^3-y^2) = 1$ and $\lambda'(x^4-y^4) = 3$.
4. The family $f_s(x,y) = (x+y^3)(x^4+y^4)(x^2+y)+sy^5$ verifies $\lambda'(f_0) = 4$ with the smooth germs $x+y^3$, x^2+y and $\lambda'(x^4+y^4) = 3$.

8. Conjectures for the jump

We give a conjectural value for $\lambda(f_0)$ in the case that $f_0 = x^p - y^q$ with $p \geqslant q$.

1. If $\gcd(p,q) = 1$ then $\lambda(f_0) = \lambda'(f_0) = 1$.
2. If $p = q$ and q is prime then $\lambda'(f_0) = q-1$, with the family $f_s(x,y) = x^q+y^q+sx^{q-1}$. And we conjecture that $\lambda(f_0) = q-2$ with the family $f_s(x,y) = x^q + y^q + s(x+y)^{q-1}$.
3. If $p = kq$ ($k > 1$) and q is prime, then $\lambda'(f_0) = q-1$, with the family $f_s(x,y) = x^p + y^q + sx^{p-1}$. It is conjectured that $\lambda(f_0) = \lambda'(f_0)$.
4. If q is not prime and $p = kq$, $k \in \mathbb{N}^*$ then let $q = ab$ with $a \geqslant 2$ the smallest prime divisor of q. Then $\lambda'(f_0) = q-1 = ab-1$ for the family $f_s(x,y) = x^p - y^q + sx^{p-1}$. It is conjectured that $\lambda(f_0) = ab-b$, which jump is reached for the family $f_s(x,y) = x^p - (y^a + sx^{ka-1})^b$.
5. If $\gcd(p,q) = d$ with $1 < d < q \leqslant p$ then $\lambda'(f_0) = d$. And it is conjectured that $\lambda(f_0) = d$ too.

We make a remark for point (4), let $g_0(x,y) = x^{p/b} - y^{q/b} = x^{ka} - y^a$. Then g_0 verifies the hypotheses of point (3) where we have conjectured $\lambda(f_0) = a-1$ for the deformation $g_s(x,y) = x^{ka} - y^a - sx^{ka-1}$. Then we calculate $g_s(x,y)^b = (x^{ka} - y^a - sx^{ka-1})^b$ which is of course not a reduced polynomial. We develop and we have an approximation of $g_s(x,y)^b$ if we set $f_s(x,y) = x^{kab} - (y^a + sx^{ka-1})^b = x^p - (y^a + sx^{ka-1})^b$ with a jump equal to $ab - b$.

Apart from the conjectures above we ask some questions. Even if it seems hard to give a formula for the jump, maybe the following is easier:

Question 1. Find an algorithm that computes λ.

Finally the problem of the jump can be seen as a weak form of the problem of adjacency. For example the list of possible Milnor numbers arising from deformations of $f_0(x,y) = x^4 - y^4$ is $(9,7,6,5,4,3,2,1,0)$. Then the gap between the first term $9 = \mu(f_0)$ and the second term is the jump $\lambda(f_0) = 2$. Then the following question is a generalization of the problem of the jump.

Question 2. Give the list of all possible Milnor numbers arising from deformations of a germ.

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