INVARIANCE OF MILNOR NUMBERS AND TOPOLOGY OF COMPLEX POLYNOMIALS

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Abstract. We give a global version of Lê-Ramanujam μ-constant theorem for polynomials. Let (f_t), $t \in [0,1]$, be a family of polynomials of n complex variables with isolated singularities, whose coefficients are polynomials in t. We consider the case where some numerical invariants are constant (the affine Milnor number $\mu(t)$, the Milnor number at infinity $\lambda(t)$, the number of critical values, the number of affine critical values, the number of critical values at infinity). Let $n = 2$, we also suppose the degree of the f_t is a constant, then the polynomials f_0 and f_1 are topologically equivalent. For $n > 3$ we suppose that critical values at infinity depend continuously on t, then we prove that the geometric monodromy representations of the f_t, are all equivalent.

1. Introduction

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geqslant 2$. By a result of Thom [Th] there is a minimal *set of critical values* $\mathcal{B}$ of point of $\mathbb{C}$ such that $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a fibration.

1.1. **Affine singularities.** We suppose that *affine singularities are isolated i.e.* that the set $\{x \in \mathbb{C}^n \mid \operatorname{grad}_f x = 0\}$ is a finite set. Let μ_c be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathit{aff}} = \big\{c \mid \mu_c > 0\big\} \quad \text{and} \quad \mu = \sum_{c \in \mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*.

1.2. **Singularities at infinity.** See [Br]. Let d be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, let $f = f^d + f^{d-1} + \cdots + f^0$ where f^j is homogeneous of degree j. Let $\bar{f}(x, x_0)$ (with $x = (x_1, \ldots, x_n)$) be the homogenization of f with the new variable x_0: $\bar{f}(x, x_0) = f^d(x) + f^{d-1}(x)x_0 + \ldots + f^0(x)x_0^d$. Let

$$X = \big\{((x : x_0), c) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x, x_0) - cx_0^d = 0\big\}.$$

Date: January 14, 2002.

Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(x_0 = 0)$. The singular locus of X has the form $\Sigma \times \mathbb{C}$ where

$$\Sigma = \left\{ (x:0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0 \right\} \subset \mathcal{H}_\infty.$$

We suppose that f has *isolated singularities at infinity* that is to say that Σ is finite. This is always true for $n = 2$. For a point $(x:0) \in \mathcal{H}_\infty$, assume, for example, that $x = (x_1, \ldots, x_{n-1}, 1)$ and set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, x_0) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - c x_0^d.$$

Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of F_c at the point $(\check{x}, 0)$. If $(x:0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic s, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many c, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Let

$$\mathcal{B}_\infty = \big\{ c \in \mathbb{C} \mid \lambda_c > 0 \big\} \quad \text{and} \quad \lambda = \sum_{c\in\mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity.* We can now describe the set of critical values $\mathcal{B}$ as follows (see [HL] and [Pa]):

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

Moreover by [HL] and [ST] for $s \notin \mathcal{B}$, $f^{-1}(s)$ has the homotopy type of a wedge of $\lambda + \mu$ spheres of real dimension $n - 1$.

1.3. Statement of the results.

Theorem 1. *Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ whose coefficients are polynomials in t. We suppose that affine singularities and singularities at infinity are isolated. Let suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{\mathit{aff}}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t \in [0,1]$. Moreover let us suppose that critical values at infinity $\mathcal{B}_\infty(t)$ depend continuously on t. Then the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_1 : f_1^{-1}(\mathbb{C} \setminus \mathcal{B}(1)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(1)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic.*

Remark 1. As a consequence for $n \neq 3$ and $* \notin \mathcal{B}(0) \cup \mathcal{B}(1)$ the monodromy representations

$$\pi_1(\mathbb{C} \setminus \mathcal{B}(0), *) \longrightarrow \mathrm{Diff}(f_0^{-1}(*)) \text{ and}$$

$$\pi_1(\mathbb{C} \setminus \mathcal{B}(1), *) \longrightarrow \mathrm{Diff}(f_1^{-1}(*))$$

are equivalent (where $\mathrm{Diff}(f_t^{-1}(*))$ denotes the diffeomorphisms of $f_t^{-1}(*)$ modulo diffeomorphisms isotopic to identity).

Remark 2. The restriction $n \neq 3$, as in [LR], is due to the use of the h-cobordism theorem.

Remark 3. This result extends a theorem of Hà H.V and Pham T.S. [HP] which deals only with monodromy at infinity (which correspond to a loop around the whole set $\mathcal{B}(t)$) for $n = 2$. For $n > 3$ the invariance of monodromy at infinity is stated by M. Tibăr in [Ti]. The proof is based on the articles of Hà H.V.-Pham T.S. [HP] and of Lê D.T.-C.P. Ramanujam [LR].

Lemma 2. *Under the hypotheses of the previous theorem (except the hypothesis of continuity of the critical values), and one of the following conditions:*

- $n = 2$*, and* $\deg f_t$ *does not depend on* t*;*
- $\deg f_t$*, and* $\Sigma(t)$ *do not depend on* t*, and for all* $(x : 0) \in \Sigma(t)$*,* $\nu_{\check{x}}(t)$ *is independent of* t*;*

we have that $\mathcal{B}_\infty(t)$ *depends continuously on* t*,* i.e. *if* $c(\tau) \in \mathcal{B}_\infty(\tau)$ *then for all* t *near* τ *there exists* $c(t)$ *near* $c(\tau)$ *such that* $c(t) \in \mathcal{B}_\infty(t)$*.*

Under the hypothesis that there is no singularity at infinity we can prove the stronger result:

Theorem 3. *Let* $(f_t)_{t\in[0,1]}$ *be a family of complex polynomials whose coefficients are polynomials in* t*. Suppose that* $\mu(t)$*,* $\#\mathcal{B}_{\mathit{aff}}(t)$ *do not depend on* $t \in [0,1]$*. Moreover suppose that* $n \neq 3$ *and for all* $t \in [0,1]$ *we have* $\mathcal{B}_\infty(t) = \varnothing$*. Then the polynomials* f_0 *and* f_1 *are topologically equivalent that is to say there exists homeomorphisms* Φ *and* Ψ *such that*

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\ \Phi\ } & \mathbb{C}^n \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathbb{C} & \xrightarrow[\ \Psi\ ]{} & \mathbb{C}. \end{array}$$

For the proof we glue the former study with the version of the μ-constant theorem of Lê D.T. and C.P. Ramanujam stated by J.G. Timourian [Tm]: a μ-constant deformation of germs of isolated hypersurface singularity is a product family.

For polynomials in two variables we can prove the following theorem which is a global version of Lê-Ramanujam-Timourian theorem:

Theorem 4. *Let* $n = 2$*. Let* $(f_t)_{t\in[0,1]}$ *be a family of complex polynomials whose coefficients are polynomials in* t*. Suppose that the integers* $\mu(t)$*,* $\lambda(t)$*,* $\#\mathcal{B}(t)$*,* $\#\mathcal{B}_{\mathit{aff}}(t)$*,* $\#\mathcal{B}_\infty(t)$*,* $\deg f_t$ *do not depend on* $t \in [0,1]$*. Then the polynomials* f_0 *and* f_1 *are topologically equivalent.*

It uses a result of L. Fourrier [Fo] that give a necessary and sufficient condition for polynomials to be topologically equivalent outside sufficiently large compact sets of $\mathbb{C}^2$.

This work was initiated by an advice of Lê D.T. concerning the article [Bo]: "It is easier to find conditions for polynomials to be equivalent than find all polynomials that respect a given condition."

We will denote $B_R = \{x \in \mathbb{C}^n \mid \|x\| \leqslant R\}$, $S_R = \partial B_R = \{x \in \mathbb{C}^n \mid \|x\| = R\}$ and $D_r(c) = \{s \in \mathbb{C} \mid \|s-c\| \leqslant r\}$.

2. Fibrations

In this paragraph we give some properties for a complex polynomial $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. The two first lemmas are consequences of transversality properties. There are direct generalizations of lemmas of [HP]. Let f be a polynomial of n complex variables with isolated affine singularities and with isolated singularities at infinity. For each fiber $f^{-1}(c)$ there is a finite number of real numbers $R > 0$ such that $f^{-1}(c)$ has non-transversal intersection with the sphere S_R. So for a sufficiently large number $R(c)$ the intersection $f^{-1}(c)$ with S_R is transversal for all $R \geqslant R(c)$. Let R_1 be the maximum of the $R(c)$ with $c \in \mathcal{B}$. We choose a small ε, $0 < \varepsilon \ll 1$ such that for all values c in the bifurcation set $\mathcal{B}$ of f and for all $s \in D_\varepsilon(c)$ the intersection $f^{-1}(s) \cap S_{R_1}$ is transversal, this is possible by continuity of the transversality. Let choose $r > 0$ such that $\mathcal{B}$ is contained in the interior of $D_r(0)$. We denote

$$K = D_r(0) \setminus \bigcup_{c \in \mathcal{B}} \mathring{D}_\varepsilon(c).$$

Lemma 5. *There exists $R_0 \gg 1$ such that for all $R \geqslant R_0$ and for all s in K, $f^{-1}(s)$ intersects S_R transversally.*

Proof. We have to adapt the beginning of the proof of [HP]. If the assertion is false then we have a sequence (x_k) of points of $\mathbb{C}^n$ such that $f(x_k) \in K$ and $\|x_k\| \to +\infty$ as $k \to +\infty$ and such that there exists complex numbers λ_k with $\operatorname{grad}_f x_k = \lambda_k x_k$, where the gradient is Milnor gradient: $\operatorname{grad}_f = \left(\overline{\frac{\partial f}{\partial x_1}}, \ldots, \overline{\frac{\partial f}{\partial x_n}}\right)$. Since K is a compact set we can suppose (after extracting a sub-sequence, if necessary) that $f(x_k) \to c \in K$ as $k \to +\infty$. Then by the Curve Selection Lemma of [NZ] there exists a real analytic curve $x :]0, \varepsilon[\longrightarrow \mathbb{C}^n$ such that $x(\tau) = a\tau^\beta + a_1\tau^{\beta+1} + \cdots$ with $\beta < 0$, $a \in \mathbb{R}^{2n} \setminus \{0\}$ and $\operatorname{grad}_f x(\tau) = \lambda(\tau)x(\tau)$. Then $f(x(\tau)) = c + c_1\tau^\rho + \cdots$ with $\rho > 0$. Then we can redo the calculus of [HP]:

$$\frac{df(x(\tau))}{d\tau} = \left\langle \frac{dx}{d\tau}, \operatorname{grad}_f x(\tau) \right\rangle = \bar{\lambda}(\tau) \left\langle \frac{dx}{d\tau}, x(\tau) \right\rangle$$

it implies

$$|\lambda(\tau)| \leqslant 2 \frac{\left|\frac{df(x(\tau))}{d\tau}\right|}{\frac{d\|x(\tau)\|^2}{d\tau}}.$$

As $\|x(\tau)\| = b_1\tau^\beta + \cdots$ with $b_1 \in \mathbb{R}_+^*$ and $\beta < 0$ we have $|\lambda(\tau)| \leqslant \gamma \frac{\tau^{\rho-1}}{\tau^{2\beta-1}} = \gamma\tau^{\rho-2\beta}$ where γ is a constant. We end the proof be using the characterization of critical value at infinity in [Pa]:

$$\|x(\tau)\|^{1-1/N} \|\operatorname{grad}_f x(\tau)\| = \|x(\tau)\|^{1-1/N} |\lambda(\tau)|\, \|x(\tau)\| \leqslant \gamma\tau^{\rho-\beta/N}$$

As $\rho > 0$ and $\beta < 0$, for all $N > 0$ we have that $\|x(\tau)\|^{1-1/N}\|\operatorname{grad}_f x(\tau)\| \to 0$ as $\tau \to 0$. By [Pa] it implies that the value c (the limit of $f(x(\tau))$ as $\tau \to 0$) is in $\mathcal{B}_\infty$. But as $c \in K$ it is impossible. □

This first lemma enables us to get the following result: because of the transversality we can find a vector field tangent to the fibers of f and pointing out the spheres S_R. Integration of such a vector field gives the next lemma.

Lemma 6. *The fibrations $f : f^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ and $f : f^{-1}(K) \longrightarrow K$ are differentiably isomorphic.*

We will also need the following fact:

Lemma 7. *The fibrations $f : f^{-1}(K) \longrightarrow K$ and $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ are differentiably isomorphic.*

The following lemma is adapted from [LR]. For completeness we give the proof.

Lemma 8. *Let R, R' with $R \geqslant R'$ be real numbers such that the intersections $f^{-1}(K) \cap S_R$ and $f^{-1}(K) \cap S_{R'}$ are transversal. Let us suppose that $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are fibrations with fibers homotopic to a wedge of ν $(n-1)$-dimensional spheres. Then the fibrations are fiber homotopy equivalent. And for $n \neq 3$ the fibrations are differentiably equivalent.*

Proof. The first part is a consequence of a result of A. Dold [Do, th. 6.3]. The first fibration is contained in the second. By the result of Dold we only have to prove that if $* \in \partial D_r$ then the inclusion of $F' = f^{-1}(*) \cap B_{R'}$ in $F = f^{-1}(*) \cap B_R$ is an homotopy equivalence. To see this we choose a generic x_0 in $\mathbb{C}^n$ such that the real function $x \mapsto \|x - x_0\|$ has non-degenerate critical points of index less than n (see [M1, §7]). Then F is obtained from F' by attaching cells of index less than n. For $n = 2$ the fibers are homotopic to a wedge of ν circles, then the inclusion of F' in F is an homotopy equivalence. For $n > 2$ the fibers F, F' are simply connected and the morphism $H_i(F') \longrightarrow H_i(F)$ induced by inclusion is an isomorphism. For $i \neq n-1$ this is obvious since F and F' have the homotopy type of a wedge of $(n-1)$-dimensional spheres, and for $i = n-1$ the exact sequence of the pair (F, F') is

$$H_n(F, F') \longrightarrow H_{n-1}(F) \longrightarrow H_{n-1}(F') \longrightarrow H_{n-1}(F, F')$$

with $H_n(F, F') = 0$, $H_{n-1}(F)$ and $H_{n-1}(F')$ free of rank ν, and $H_{n-1}(F, F')$ torsion-free. Then the inclusion of F' in F is an homotopy equivalence.

The second part is based on the h-cobordism theorem. Let $X = f^{-1}(K) \cap B_R \setminus \mathring{B}_{R'}$, then as f has no affine critical points in X (because there is no critical values in K) and f is transversal to $f^{-1}(K) \cap S_R$ and to $f^{-1}(K) \cap S_{R'}$ then by Ehresmann theorem $f : X \longrightarrow K$ is a fibration. We denote $F \setminus \mathring{F}'$ by F^*. We get an isomorphism $H_i(\partial F') \longrightarrow H_i(F^*)$ for all i because $H_i(F^*, \partial F') = H_i(F, F') = 0$. For $n = 2$ it implies that F^* is diffeomorphic to a product $[0, 1] \times \partial F'$. For $n > 3$ we will use the

h-cobordism theorem to F^* to prove this. We have $\partial F^* = \partial F' \cup \partial F$; $\partial F'$ and ∂F are simply connected: if we look at the function $x \mapsto -\|x - x_0\|$ on $f^{-1}(*)$ for a generic x_0, then $F = f^{-1}(*) \cap B_R$ and $F' = f^{-1}(*) \cap B_{R'}$ are obtained by gluing cells of index more or equal to $n-1$. So their boundary is simply connected. For a similar reason F^* is simply connected. As we have isomorphisms $H_i(\partial F') \longrightarrow H_i(F^*)$ and both spaces are simply connected then by Hurewicz-Whitehead theorem the inclusion of $\partial F'$ in F^* is an homotopy equivalence. Now F^*, $\partial F'$, ∂F are simply connected, the inclusion of $\partial F'$ in F^* is an homotopy equivalence and F^* has real dimension $2n-2 \geqslant 6$. So by the h-cobordism theorem [M2] F^* is diffeomorphic to the product $[0,1] \times \partial F'$. Then the fibration $f : X \longrightarrow K$ is differentiably equivalent to the fibration $f : [0,1] \times (f^{-1}(K) \cap S_{R'}) \longrightarrow K$ so the fibrations $f : f^{-1}(K) \cap B_{R'} \longrightarrow K$ and $f : f^{-1}(K) \cap B_R \longrightarrow K$ are differentiably equivalent. □

3. Family of polynomials

Let $(f_t)_{t\in[0,1]}$ be a family of polynomials that verify hypotheses of theorem 1.

Lemma 9 ([HP]). *There exists $R \gg 1$ such that for all $t \in [0,1]$ the affine critical points of f_t are in $\mathring{B}_R$.*

Proof. It is enough to prove it on $[0,\tau]$ with $\tau > 0$. We choose $R \gg 1$ such that all the affine critical points of f_0 are in $\mathring{B}_R$. We denote

$$\phi_t = \frac{\operatorname{grad}_{f_t}}{\| \operatorname{grad}_{f_t} \|} : S_R \longrightarrow S_1.$$

Then $\deg \phi_0 = \mu(0)$. For all $x \in S_R$, $\operatorname{grad}_{f_0} x \neq 0$, and by continuity there exist $\tau > 0$ such that for $t \in [0,\tau]$ and all $x \in S_R$, $\operatorname{grad}_{f_t} x \neq 0$. Then the maps ϕ_t are homotopic (the homotopy is $\phi : S_R \times [0,\tau] \longrightarrow S_1$ with $\phi(x,t) = \phi_t(x)$). And then $\mu(0) = \deg \phi_0 = \deg \phi_t \leqslant \mu(t)$. If there exists a family $x(t) \in \mathbb{C}^n$ of affine critical points of ϕ_t such that $\|x(t)\| \to +\infty$ as $t \to 0$, then for a sufficiently small t, $x(t) \notin B_R$ and then $\mu(t) > \deg \phi_t$. It contradicts the hypothesis $\mu(0) = \mu(t)$. □

Lemma 10. *There exists $r \gg 1$ such that the subset $\big\{(c,t) \in D_r(0) \times [0,1] \mid c \in \mathcal{B}(t)\big\}$ is a braid of $D_r(0) \times [0,1]$.*

It enables us to choose $* \in \partial D_r(0)$ which is a regular value for all f_t, $t \in [0,1]$. In other words if we enumerate $\mathcal{B}(0)$ as $\{c_1(0), \ldots, c_m(0)\}$ then there is continuous functions $c_i : [0,1] \longrightarrow D_r(0)$ such that for $i \neq j$, $c_i(t) \neq c_j(t)$. This enables us to identify $\pi_1(\mathbb{C} \setminus \mathcal{B}(0), *)$ and $\pi_1(\mathbb{C} \setminus \mathcal{B}(1), *)$.

Proof. Let τ be in $[0,1]$ and $c(\tau)$ be a critical value of f_τ then for all t near τ there exists a critical value $c(t)$ of f_t. It is an hypothesis for the critical values at infinity and this fact is well-known for affine critical values as the coefficients of f_t are smooth functions of t, see for example [Br, Prop. 2.1].

Moreover by the former lemma there can not exist critical values that escape at infinity *i.e.* a $\tau \in [0,1]$ such that $|c(t)| \to +\infty$ as $t \to \tau$. For affine critical values it is a consequence of the former lemma (or we can make the same proof as we now will perform for the critical values at infinity). For $\mathcal{B}_\infty(t)$ let us suppose that there is critical values that escape at infinity. By continuity of the critical values at infinity with respect to t we can suppose that there is a continuous function $c_0(t)$ on $]0,\tau]$ ($\tau > 0$) with $c_0(t) \in \mathcal{B}_\infty(t)$ and $|c(t)| \to +\infty$ as $t \to 0$. By continuity of the critical values at infinity, if $\mathcal{B}_\infty(0) = \{c_1(0), \ldots, c_p(0)\}$ there exist continuous functions $c_i(t)$ on $[0,\tau]$ such that $c_i(t) \in \mathcal{B}_\infty(t)$ for all $i = 1, \ldots, p$. And for a sufficiently small $t > 0$, $c_0(t) \neq c_i(t)$ ($i = 1, \ldots, p$) then $\#\mathcal{B}_\infty(0) < \#\mathcal{B}_\infty(t)$ which contradicts the constancy of $\#\mathcal{B}_\infty(t)$.

Finally there can not exist ramification points: suppose that there is a τ such that $c_i(\tau) = c_j(\tau)$ (and $c_i(t), c_j(t)$ are not equal in a neighborhood of τ). Then if $c_i(\tau) \in \mathcal{B}_{\mathit{aff}}(\tau) \setminus \mathcal{B}_\infty(\tau)$ (*resp.* $\mathcal{B}_\infty(\tau) \setminus \mathcal{B}_{\mathit{aff}}(\tau)$, $\mathcal{B}_\infty(\tau) \cap \mathcal{B}_{\mathit{aff}}(\tau)$) there is jump in $\#\mathcal{B}_{\mathit{aff}}(t)$ (*resp.* $\#\mathcal{B}_\infty(t)$, $\#\mathcal{B}(t)$) near τ which is impossible by assumption. □

Let $R_0, K, D_r(0), D_\varepsilon(c)$ be the objects of the former section for the polynomial $f = f_0$. Moreover we suppose that R_0 is greater than the R obtained in lemma 9.

Lemma 11. *There exists $\tau \in]0,1]$ such that for all $t \in [0,\tau]$ we have the properties:*

- *$c_i(t) \in D_\varepsilon(c_i(0))$, $i = 1, \ldots, m$;*
- *for all $s \in K$, $f_t^{-1}(s)$ intersects S_{R_0} transversally.*

Proof. The first point is just the continuity of the critical values $c_i(t)$. The second point is the continuity of transversality: if the property is false then there exists sequences $t_k \to 0$, $x_k \in S_{R_0}$ and $\lambda_k \in \mathbb{C}$ such that $\operatorname{grad}_{f_{t_k}} x_k = \lambda_k x_k$. We can suppose that (x_k) converges (after extraction of a sub-sequence, if necessary). Then $x_k \to x \in S_{R_0}$, $\operatorname{grad}_{f_{t_k}} x_k \to \operatorname{grad}_{f_0} x$, and $\lambda_k = \langle \operatorname{grad}_{f_{t_k}} x_k \,|\, x_k \rangle / \|x_k\|^2 = \langle \operatorname{grad}_{f_{t_k}} x_k \mid x_k \rangle / {R_0}^2$ converges toward $\lambda \in \mathbb{C}$. Then $\operatorname{grad}_{f_0} x = \lambda x$ and the intersection is non-transversal. □

Lemma 12. *The fibrations $f_0 : f_0^{-1}(K) \cap B_{R_0} \longrightarrow K$ and $f_\tau : f_\tau^{-1}(K) \cap B_{R_0} \longrightarrow K$ are differentiably isomorphic.*

Proof. Let

$$F : \mathbb{C}^n \times [0,1] \longrightarrow \mathbb{C} \times [0,1], \qquad (x,t) \mapsto (f_t(x), t).$$

We want to prove that the fibrations $F_0 : \Sigma_0 = F^{-1}(K \times \{0\}) \cap (B_{R_0} \times \{0\}) \longrightarrow K$, $(x,0) \mapsto f_0(x)$ and $F_\tau : \Sigma_\tau = F^{-1}(K \times \{\tau\}) \cap (B_{R_\tau} \times \{\tau\}) \longrightarrow K$, $(x,\tau) \mapsto f_\tau(x)$ are differentiably isomorphic. Let denote $[0,\tau]$ by I. Then F has maximal rank on $F^{-1}(K \times I) \cap (\mathring{B}_{R_0} \times I)$ and on the boundary $F^{-1}(K \times I) \cap (S_{R_0} \times I)$. By Ehresmann theorem $F : F^{-1}(K \times I) \cap (B_{R_0} \times I) \longrightarrow K \times I$ is a fibration. But we can not argue as in [LR] since the restriction of F on the set $\big\{(x,t) \in S_{R_0} \times I \mid f_t(x) \in D_r(0)\big\}$ is not a trivial fibration.

As in [HP] we build a vector field that give us a diffeomorphism between the two fibrations F_0 and F_τ. Let R_2 be a real number close to R_0 such that $R_2 < R_0$. On the set $F^{-1}(K \times I) \cap (\cup_{R_2<R<R_0} S_R \times I)$ we build a vector field v_1 such that for $z \in S_R \times I$ $(R_2 < R < R_0)$, $v_1(z)$ is tangent to $S_R \times I$ and we have $d_zF.v_1(z) = (0,1)$. On the set $F^{-1}(K \times I) \cap (\mathring{B}_{R_3} \times I)$ with $R_2 < R_3 < R_0$ we build a second vector field v_2 such that $d_zF.v_2(z) = (0,1)$, this is possible because F is a submersion on this set.

By gluing these vector fields v_1 and v_2 by a partition of unity and by integrating the corresponding vector field we obtain integral curves $p_z : \mathbb{R} \longrightarrow F^{-1}(K \times I) \cap B_{R_0} \times I$ for $z \in \Sigma_0$ such that $p_z(0) = z$ and $p_z(\tau) \in \Sigma_\tau$. It induces a diffeomorphism $\Phi : \Sigma_0 \longrightarrow \Sigma_\tau$ such that $F_0 = F_\tau \circ \Phi$; that makes the fibrations isomorphic. □

Proof of theorem 1. It suffices to prove the theorem for an interval $[0,\tau]$ with $\tau > 0$. We choose τ as in lemma 11. By lemma 7, $f_0 : f^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_0 : f_0^{-1}(K) \longrightarrow K$ are differentiably isomorphic fibrations. Then by lemma 6, the fibration $f_0 : f_0^{-1}(K) \longrightarrow K$ is differentiably isomorphic to $f_0 : f_0^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$ which is, by lemma 12 differentiably isomorphic to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_{R_0} \longrightarrow K$.

By continuity of transversality (lemma 11) $f_\tau^{-1}(K)$ has transversal intersection with S_{R_0}, we choose a large real number R (by lemma 5 applied to f_τ) such that $f_\tau^{-1}(K)$ intersects S_R transversally. The last fibration is fiber homotopy equivalent to $f_\tau : f_\tau^{-1}(K) \cap \mathring{B}_R \longrightarrow K$: it is the first part of lemma 8 because the fiber $f_\tau^{-1}(*) \cap \mathring{B}_{R_0}$ is homotopic to a wedge of $\mu(0) + \lambda(0)$ circles and the fiber $f_\tau^{-1}(*) \cap \mathring{B}_R$ is homotopic to a wedge of $\mu(\tau) + \lambda(\tau)$ circles; as $\mu(0) + \lambda(0) = \mu(\tau) + \lambda(\tau)$ we get the desired conclusion. Moreover for $n \neq 3$ by the second part of lemma 8 the fibrations are differentiably isomorphic.

Applying lemma 6 and 7 to f_τ this fibration is differentiably isomorphic to $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$. As a conclusion the fibrations $f_0 : f_0^{-1}(\mathbb{C} \setminus \mathcal{B}(0)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(0)$ and $f_\tau : f_\tau^{-1}(\mathbb{C} \setminus \mathcal{B}(\tau)) \longrightarrow \mathbb{C} \setminus \mathcal{B}(\tau)$ are fiber homotopy equivalent, and for $n \neq 3$ are differentiably isomorphic □

4. Around affine singularities

We now work with $t \in [0,1]$. We suppose in this paragraph that the critical values $\mathcal{B}(t)$ depend analytically on $t \in [0,1]$. This enables us to construct a diffeomorphism:

$$\chi : \mathbb{C} \times [0,1] \longrightarrow \mathbb{C} \times [0,1], \qquad \text{with } \chi(x,t) = (\chi_t(x), t),$$

such that $\chi_0 = \mathrm{id}$ and $\chi_t(\mathcal{B}(t)) = \mathcal{B}(0)$. We denote χ_1 by Ψ, so that $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ verify $\Psi(\mathcal{B}(1)) = \mathcal{B}(0)$. Moreover we can suppose that χ_t is equal to id on $\mathbb{C} \setminus D_r(0)$ this is possible because for all $t \in [0,1]$, $\mathcal{B}(t) \subset D_r(0)$. Finally χ defines a vector field w of $\mathbb{C} \times [0,1]$ by $\frac{\partial \chi}{\partial t}$.

We need a non-splitting of the affine singularity, this principle has been proved by C. Has Bey ([HB], $n = 2$) and by F. Lazzeri ([La], for all n).

Lemma 13. *Let $x(\tau)$ be an affine singular point of f_τ and let U_τ be an open neighborhood of $x(\tau)$ in $\mathbb{C}^n$ such that $x(\tau)$ is the only affine singular point of f_τ in U_τ. Suppose that for all t closed to τ, the restriction of f_t to U_τ has only one critical value. Then for all t sufficiently closed to τ, there is one, and only one, affine singular point of f_t contained in U_τ.*

So we can enumerate the singularities: if we denote the affine singularities of f_0 by $\{x_i(0)\}_{i\in J}$ then there is continuous functions $x_i : [0,1] \longrightarrow \mathbb{C}^n$ such that $\{x_i(t)\}_{i\in J}$ is the set of affine singularities of f_t. Let us notice that there can be two distinct singular points of f_t with the same critical value. We suppose that (f_t) verifies the hypotheses of theorem 1, that $n \neq 3$, and $\mathcal{B}(t)$ depends analytically on t. This and the former lemma imply that for all $t \in [0,1]$ the local Milnor number of f_t at $x(t)$ is equal to the local Milnor number of f_0 at $x(0)$. The improved version of Lê-Ramanujam theorem by J.G. Timourian [Tm] for a family of germs with constant local Milnor numbers proves that (f_t) is locally a product family.

Theorem 14 (Lê-Ramanujam-Timourian)**.** *Let $x(t)$ be a singular points of f_t. There exists U_t, V_t neighborhoods of $x(t)$, $f_t(x(t))$ respectively and an homeomorphism Ω^{in} such that if $U = \bigcup_{t\in[0,1]} U_t \times \{t\}$ and $V = \bigcup_{t\in[0,1]} V_t \times \{t\}$ the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{\Omega^{\text{in}}} & U_0 \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times \text{id}} \\ V & \xrightarrow[\chi]{} & V_0 \times [0,1]. \end{array}$$

In particular it proves that the polynomials f_0 and f_1 are locally topologically equivalent: we get an homeomorphism Φ_{in} such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{\Phi_{\text{in}}} & U_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ V_1 & \xrightarrow[\Psi]{} & V_0. \end{array}$$

By lemma 9 we know that for all $t \in [0,1]$, $\mathcal{B}(t) \subset D_r(0)$. We extend the definition of R_0 and R_1 to all f_t. Be continuity of transversality and compactness of $[0,1]$ we choose R_1 such that

$$\forall c \in \mathcal{B}(0) \quad \forall R \geqslant R_1 \quad f_0^{-1}(c) \pitchfork S_R \quad \text{and} \quad \forall t \in [0,1] \quad \forall c \in \mathcal{B}(t) \quad f_t^{-1}(c) \pitchfork S_{R_1}.$$

For a sufficiently small ε we denote

$$K(0) = D_r(0) \setminus \bigcup_{c \in \mathcal{B}_\infty(0)} D_\varepsilon(c), \qquad K(t) = \chi_t^{-1}(K(0))$$

and we choose $R_0 \geqslant R_1$ such that

$\forall s \in K(t) \quad \forall R \geqslant R_0 \quad f_0^{-1}(s) \pitchfork S_R \quad$ and $\quad \forall t \in [0,1] \quad \forall s \in K(t) \quad f_t^{-1}(s) \pitchfork S_{R_0}$.

We denote

$$B'_t = B_{R_1} \cup \left(f_t^{-1}(K(t)) \cap \mathring{B}_{R_0}\right), \quad t \in [0,1].$$

Lemma 15. *There exists an homeomorphism Φ such that we have the commutative diagram:*

$$\begin{array}{ccc} B'_1 & \xrightarrow{\Phi} & B'_0 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ D_r(0) & \xrightarrow[\Psi]{} & D_r(0). \end{array}$$

Proof. We denote by U'_t a neighborhood of $x(t)$ such that $\bar{U'_t} \subset U_t$. We denote by $\mathcal{U}_t$ (resp. $\mathcal{U}'_t$), the union (on the affine singular points of f_t) of the U_t (resp. U'_t). We set

$$B''_t = B'_t \setminus \mathcal{U}'_t, \quad t \in [0,1].$$

We can extend the homeomorphism Φ of lemma 12 to $\Phi_{\mathrm{out}} : B''_1 \longrightarrow B''_0$. We just have to extend the vector field of lemma 12 to a new vector field denoted by v' such that

- v' is tangent to $\partial \mathcal{U}'_t$,
- v' is tangent to $S_{R_1} \times [0,1]$ on $F^{-1}(D_r(0) \setminus K(t) \times \{t\})$ for all $t \in [0,1]$,
- v' is tangent to $S_{R_0} \times [0,1]$ on $F^{-1}(K(t) \times \{t\})$ for all t.
- $d_z F.v'(z) = w(F(z))$ for all $z \in \bigcup_{t \in [0,1]} B''_t \times \{t\}$, which means that Φ_{out} respect the fibrations.

If we set $B'' = \bigcup_{t \in [0,1]} B''_t \times \{t\}$ the integration of v' gives Ω_{out} and Φ_{out} such that:

$$\begin{array}{ccc} B'' & \xrightarrow{\Omega^{\mathrm{out}}} & B''_0 \times [0,1] \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle f_0 \times \mathrm{id}} \\ \mathbb{C} \times [0,1] & \xrightarrow[\chi]{} & \mathbb{C} \times [0,1], \end{array} \qquad \begin{array}{ccc} B''_1 & \xrightarrow{\Phi_{\mathrm{out}}} & B''_0 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ D_r(0) & \xrightarrow[\Psi]{} & D_r(0). \end{array}$$

We now explain how to glue Φ_{in} and Φ_{out} together. We can suppose that there exists spheres S_t centered at the singularities $x(t)$ such that if $S = \bigcup_{t \in [0,1]} S_t \times \{t\}$ $\Omega^{\mathrm{in}} : S \longrightarrow S_0 \times [0,1]$ and $\Omega^{\mathrm{out}} : S \longrightarrow S_0 \times [0,1]$. It defines $\Omega_t^{\mathrm{in}} : S_t \longrightarrow S_0$ and $\Omega_t^{\mathrm{out}} : S_t \longrightarrow S_0$. Now we define

$$\Theta_t : S_1 \longrightarrow S_0, \quad \Theta_t = \Omega_t^{\mathrm{in}} \circ (\Omega_t^{\mathrm{out}})^{-1} \circ \Phi_{\mathrm{out}}.$$

Then $\Theta_0 = \Phi_{\text{out}}$ and $\Theta_1 = \Phi_{\text{in}}$. On a set homeomophic to $S \times [0,1]$ included in $\bigcup_{t\in[0,1]} U_t \setminus U'_t$ we glue Φ_{in} to Φ_{out}, moreover this gluing respect the fibrations f_0 and f_1. We end by doing this construction for all affine singular points. $\square$

Proof of theorem 3. We firstly prove that affine critical values are analytic functions of t. Let $c(0) \in \mathcal{B}_{\mathit{aff}}(0)$, the set $\big\{(c(t),t) \mid t \in [0,1]\big\}$ is a real algebraic subset of $\mathbb{C}\times[0,1]$ as all affine critical points are contained in B_{R_0} (lemma 9). In fact there is a polynomial $P \in \mathbb{C}[x,t]$ such that $(c=0)$ is equal to $(P=0)\cap\mathbb{C}\times[0,1]$. Because the set of critical values is a braid of $\mathbb{C}\times[0,1]$ (lemma 10) then $c:[0,1]\longrightarrow\mathbb{C}$ is a smooth analytic function.

If we suppose that $\mathcal{B}_\infty(t) = \varnothing$ for all $t \in [0,1]$ then by lemma 6 we can extend $\Phi : B'_1 \longrightarrow B'_0$ to $\Phi : f_1^{-1}(D_r(0)) \longrightarrow f_0^{-1}(D_r(0))$. And as $\mathcal{B}(t) \subset D_r(0)$ by a lemma similar to lemma 7 we can extend the homeomorphism to the whole space. $\square$

5. Polynomials in two variables

We set $n = 2$. Let $f_t : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that the coefficient of this family are algebraic in t. We suppose that the integers $\mu(t)$, $\lambda(t)$, $\#\mathcal{B}(t)$, $\#\mathcal{B}_{\mathit{aff}}(t)$, $\#\mathcal{B}_\infty(t)$ do not depend on $t\in[0,1]$. We also suppose the $\deg f_t$ does not depend on t.

We recall a result of L. Fourrier [Fo]. Let $f:\mathbb{C}^2\longrightarrow\mathbb{C}$ with set of critical values at infinity $\mathcal{B}_\infty$. Let $*\notin\mathcal{B}$ and $Z = f^{-1}(*)\cup\bigcup_{c\in\mathcal{B}_\infty} f^{-1}(c)$. The *total link of f* is $L_f = Z\cap S_R$ for a sufficiently large R. To f we associate a resolution $\phi:\Sigma\longrightarrow\mathbb{P}^1$, the components of the divisor of this resolution on which ϕ is surjective are the *dicritical components*. For each dicritical component D we have a branched covering $\phi : D\longrightarrow\mathbb{P}^1$. If the set of dicritical components is D_{dic} we then have the restriction of ϕ, $\phi_{\text{dic}} : D_{\text{dic}}\longrightarrow\mathbb{P}^1$. The 0*-monodromy representation* is the representation

$$\pi_1(\mathbb{C}\setminus\mathcal{B}) \longrightarrow \operatorname{Aut}\big(\phi_{\text{dic}}^{-1}(*)\big).$$

Theorem 16 (Fourrier)**.** *Let f,g be complex polynomials in two variables with equivalent* 0*-monodromy representations and equivalent total links then there exist homeomorphisms Φ_∞ and Ψ_∞ and compact sets C, C' of $\mathbb{C}^2$ that make the diagram commuting:*

$$\begin{array}{ccc} \mathbb{C}^2\setminus C & \xrightarrow{\Phi_\infty} & \mathbb{C}^2\setminus C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\Psi_\infty]{} & \mathbb{C}. \end{array}$$

For our family (f_t), by theorem 1 we know that the geometric monodromy representations are all equivalent, it implies that all the 0-monodromy representations of (f_t) are equivalent. Moreover if we suppose that for any $t,t'\in[0,1]$ the total links L_{f_t} and $L_{f_{t'}}$ are equivalent, then by the former theorem the polynomials f_t and $f_{t'}$ are topologically equivalent out of some compact sets of $\mathbb{C}^2$. We need a result a bit stronger which can be proved by similar arguments than in [Fo] and we will omit the proof:

Lemma 17. *Let $(f_t)_{t\in[0,1]}$ be a polynomial family such that the coefficients are algebraic functions of t. We suppose that the 0-monodromy representations and the total links are all equivalent. Then there exists compact sets $C(t)$ of $\mathbb{C}^2$ and an homeomorphism Ω^∞ such that if $\mathcal{C} = \bigcup_{t\in[0,1]} C(t) \times \{t\}$ we have a commutative diagram:*

$$\begin{array}{ccc} \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} & \xrightarrow{\Omega^\infty} & \mathbb{C}^2 \setminus C(0) \times [0,1] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times \mathrm{id}} \\ \mathbb{C} \times [0,1] & \xrightarrow[\chi]{} & \mathbb{C} \times [0,1]. \end{array}$$

We now prove a strong version of the continuity of critical values.

Lemma 18. *The critical values are smooth analytic functions of t. Moreover for $c(t) \in \mathcal{B}(t)$, the integer $\mu_{c(t)}$ and $\lambda_{c(t)}$ do not depend on $t \in [0,1]$.*

Proof. For affine critical values, refer to the proof of theorem 3. The constancy of $\mu_{c(t)}$ is a consequence of lemma 9 and lemma 13. For critical values at infinity we need a result of [Ha] and [HP] that enables to calculate critical values and Milnor numbers at infinity. As $\deg f_t$ is constant we can suppose that this degree is $\deg_y f_t$. Let denote $\Delta(x,s,t)$ the discriminant $\mathrm{Disc}_y(f_t(x,y)-s)$ with respect to y. We write

$$\Delta(x,s,t) = q_1(s,t)x^{k(t)} + q_2(s,t)x^{k(t)-1} + \cdots$$

First of all Δ has constant degree $k(t)$ in x because $k(t) = \mu(t) + \lambda(t) + \deg f_t - 1$ (see [HP]). Secondly by [Ha] we have

$$\mathcal{B}_\infty(t) = \{ s \mid q_1(s,t) = 0 \}$$

then we see that critical values at infinity depend continuously on t and that critical values at infinity are a real algebraic subset of $\mathbb{C}\times[0,1]$. For the analicity we end as in the proof of theorem 3. Finally, for a fixed t, we have that $\lambda_c = k(t) - \deg_x \Delta(x,c,t)$. In other words $q_i(t,c)$ is zero for $i = 1,\ldots,\lambda_c$ and non-zero for $i = \lambda_c + 1$. For $c(t) \in \mathcal{B}_\infty(t)$ we now prove that $\lambda_{c(t)}$ is constant. The former formula proves that $\lambda_{c(t)}$ is constant except for all but finitely many $\tau \in [0,1]$ for which $\lambda_{c(\tau)} \geqslant \lambda_{c(t)}$. But if $\lambda_{c(\tau)} > \lambda_{c(t)}$ then $\lambda(\tau) = \sum_{c\in\mathcal{B}_\infty(\tau)} \lambda_c > \sum_{c\in\mathcal{B}_\infty(t)} \lambda_c = \lambda(t)$ which contradicts the hypotheses. □

To apply lemma 17 we need to prove:

Lemma 19. *For any $t, t' \in [0,1]$ the total links L_{f_t} and $L_{f'_t}$ are equivalent.*

Proof. The problem is similar to the one of [LR] and to lemma 8. For a value $c(t)$ in $\mathcal{B}_\infty(t)$ or equal to $*$, we have that the link at infinity $f_0^{-1}(c(0)) \cap S_{R_1}$ is equivalent to the link $f_1^{-1}(c(1)) \cap S_{R_1}$ (lemma 15). But $f_1^{-1}(c(1)) \cap S_{R_1}$ is not necessarily the link at infinity for $f_1^{-1}(c(1))$. We now prove this fact; let denote $c = c(1)$. Let $R_2 \geqslant R_1$ such that for all $R \geqslant R_2$, $f_1^{-1}(c) \pitchfork S_R$, then $f_1^{-1}(c) \cap S_{R_2}$ is the link at infinity of $f_1^{-1}(c)$. We choose η, $0 < \eta \ll 1$ such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_1} and S_{R_2} and such that $f_1^{-1}(\partial D_\eta(c))$ has transversal intersection with all S_R, $R \in [R_1, R_2]$. Notice that η is much smaller than the ε of the former paragraphs and that $f_1^{-1}(s) \cap S_{R_2}$ is **not** the link at infinity of $f_1^{-1}(s)$ for $s \in \partial D_\eta(c)$. We fix R_0 smaller than R_1 such that $f_1^{-1}(D_\eta(c))$ has transversal intersection with S_{R_0}. We denote $f_1^{-1}(D_\eta(c)) \cap B_{R_i} \setminus \mathring{B}_{R_0}$ by $\mathcal{A}_i$, $i = 1, 2$. The proof is now similar to the one of lemma 8. Let A_1 and A_2 be connected components of $\mathcal{A}_1$ and $\mathcal{A}_2$ with $A_1 \subset A_2$. By Ehresmann theorem, we have fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$, $f_1 : A_2 \longrightarrow D_\eta(c)$. From one hand $f_1^{-1}(c) \cap B_{R_1}$ has the homotopy type of a wedge of $\mu + \lambda - \mu_{c(0)} - \lambda_{c(0)}$ circles, because $f_1^{-1}(c) \cap B_{R_1}$ is diffeomorphic to $f_1^{-1}(c(0)) \cap B_{R_1}$ with Euler characteristic $1 - \mu - \lambda + \mu_{c(0)} + \lambda_{c(0)}$ by Suzuki formula. From the other hand $f_1^{-1}(c) \cap B_{R_2}$ has the homotopy type of a wedge of $\mu + \lambda - \mu_{c(1)} - \lambda_{c(1)}$ circles by Suzuki formula. By lemma 18 we have that $\mu_{c(0)} + \lambda_{c(0)} = \mu_{c(1)} + \lambda_{c(1)}$, with $c = c(1)$, so the fiber $f_1^{-1}(c) \cap B_{R_1}$ and $f_1^{-1}(c) \cap B_{R_2}$ are homotopic, it implies that the fibrations $f_1 : A_1 \longrightarrow D_\eta(c)$ and $f_1 : A_2 \longrightarrow D_\eta(c)$ are fiber homotopy equivalent, and even more are diffeomorphic. It provides a diffeomorphism $\Xi : A_1 \cap S_{R_1} = A_2 \cap S_{R_1} \longrightarrow A_2 \cap S_{R_2}$ and we can suppose that $\Xi(f_1^{-1}(c) \cap A_1 \cap S_{R_1})$ is equal to $f_1^{-1}(c) \cap A_1 \cap S_{R_1}$. By doing this for all connected components of $\mathcal{A}_1$, $\mathcal{A}_2$, for all values $c \in \mathcal{B}_\infty(1) \cup \{*\}$ and by extending Ξ to the whole spheres we get a diffeomorphism $\Xi : S_{R_1} \longrightarrow S_{R_2}$ such that $\Xi(f_1^{-1}(c) \cap S_{R_1}) = f_1^{-1}(c) \cap S_{R_2}$ for all $c \in \mathcal{B}_\infty(1) \cup \{*\}$. Then the total link for f_0 and f_1 are equivalent. $\square$

Proof of theorem 4. By lemma 17 we have a trivialization $\Omega^\infty : \mathbb{C}^2 \times [0,1] \setminus \mathcal{C} \longrightarrow \mathbb{C}^2 \setminus C(0) \times [0,1]$. We can choose the R_1 (before lemma 15) such that $\mathring{C}(t) \subset B_{R_1}$. And then the proof of this lemma gives us an $\Omega^{\text{out}} : \bigcup_{t \in [0,1]} B''(t) \times \{t\} \longrightarrow B''(0) \times [0,1]$. By gluing Ω^{out} and Ω^∞ as in this proof we obtain $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathbb{C} & \xrightarrow[\Psi]{} & \mathbb{C}. \end{array}$$

Then f_0 and f_1 are topologically equivalent. $\square$

6. Continuity of the critical values at infinity

Lemma 20. *Let $(f_t)_{t \in [0,1]}$ be a family of polynomials such that the coefficients are polynomials in t. We suppose that:*

- *the total affine Milnor number $\mu(t)$ is constant;*
- *the degree $\deg f_t$ is a constant;*
- *the set of critical points at infinity $\Sigma(t)$ is finite and does not vary: $\Sigma(t) = \Sigma$;*

- *for all $(x:0) \in \Sigma$, the generic Milnor number $\nu_{\check{x}}(t)$ is independent of t.*

Then the critical values at infinity depend continuously on t, i.e. *if $c(t_0) \in \mathcal{B}_\infty(t_0)$ then for all t near t_0 there exists $c(t)$ near $c(t_0)$ such that $c(t) \in \mathcal{B}_\infty(t)$.*

Let f be a polynomial. For $x \in \mathbb{C}^n$ we have $(x:1)$ in $\mathbb{P}^n$ and if $x_n \neq 0$ we divide x by x_n to obtain local coordinates at infinity $(\check{x}', x_0)$. The following lemma explains the link between the critical points of f and those of F_c. It uses Euler relation for the homogeneous polynomial of f of degree d.

Lemma 21.

- *F_c has a critical point $(\check{x}', x_0)$ with $x_0 \neq 0$ of critical value 0 if and only if f has a critical point x with critical value c.*
- *F_c has a critical point $(\check{x}', 0)$ of critical value 0 if and only if $(x:0) \in \Sigma$.*

Proof of lemma 20. We suppose that critical values at infinity are *not* continuous functions of t. Then there exists (t_0, c_0) such that $c_0 \in \mathcal{B}_\infty(t_0)$ and for all (t,c) in a neighborhood of (t_0, c_0), we have $c \notin \mathcal{B}_\infty(t)$. Let P be the point of irregularity at infinity for (t_0, c_0). Then $\mu_P(F_{t_0,c_0}) > \mu_P(F_{t_0,c})$ $(c \neq c_0)$ by definition of $c_0 \in \mathcal{B}_\infty(t_0)$ and by semi-continuity of the local Milnor number at P we have $\nu_P(t_0) = \mu_P(F_{t_0,c}) \geqslant \mu_P(F_{t,c}) = \nu_P(t)$, $(t,c) \neq (t_0, c_0)$.

We consider t as a complex parameter. By continuity of the critical points and by conservation of the Milnor number for $(t,c) \neq (t_0,c_0)$ we have critical points $M(t,c)$ near P of $F_{t,c}$ that are not equal to P. This fact uses that $\deg f_t$ is a constant, in order to prove that $F_{t,c}$ depends continuously of t.

Let denote by V' the algebraic variety of $\mathbb{C}^3 \times \mathbb{C}^n$ defined by $(t,c,s,x) \in V'$ if and only if $F_{t,c}$ has a critical point x with critical value s (the equations are $\operatorname{grad} F_{t,c}(x) = 0, F_{t,c}(x) = s$). If $\mu_P(F_{t,c}) > 0$ for a generic (t,c) then $\left\{(t,c,0,P) \mid (t,c) \in \mathbb{C}^2\right\}$ is a subvariety of V'. We define V to be the closure of V' minus this subvariety. Then for a generic (t,c), $(t,c,0,P) \notin V$. We call $\pi : \mathbb{C}^3 \times \mathbb{C}^n \longrightarrow \mathbb{C}^3$ the projection on the first factor. We set $W = \pi(V)$. Then W is locally an algebraic variety around $(t_0, c_0, 0)$. For each (t,c) there is a non-zero finite number of values s such that $(t,c,s) \in W$. So W is locally an equi-dimensional variety of codimension 1. Then it is a germ of hypersurface of $\mathbb{C}^3$. Let $P(t,c,s)$ be the polynomial that defines W locally. We set $Q(t,c) = P(t,c,0)$. As $Q(t_0,c_0) = 0$ then in all neighborhoods of (t_0,c_0) there exists $(t,c) \neq (t_0,c_0)$ such that $Q(t,c) = 0$. Moreover there are solutions for t a real number near t_0.

Then for $(t,c) \neq (t_0,c_0)$ we have that: $Q(t,c) = 0$ if and only if $F_{t,c}$ has a critical point $M(t,c) \neq P$ with critical value 0. The point $M(t,c)$ is not equal to P because as $t \neq t_0$, $(t,c,0,P) \notin V$: it uses that $c \notin \mathcal{B}_\infty(t)$ for $t \neq t_0$, and that $\nu_P(t) = \nu_P(t_0)$. Let us notice that $M(t,c) \to P$ as $(t,c) \to (t_0,c_0)$.

We end the proof be studying the different cases:

- if we have $M(t,c)$ in $\mathcal{H}_\infty$ (of equation $(x_0 = 0)$) then $M(t,c) \in \Sigma$ which provides a contradiction because then it is equal to P;

- if we have points $M(t,c)$, not in $\mathcal{H}_\infty$, with $t \neq t_0$ then there are affine critical points $M'(t,c)$ of f_t (lemma 21), and as $M(t,c)$ tends towards P (as (t,c) tends towards (t_0,c_0)) we have that $M'(t,c)$ escapes at infinity, it contradicts the fact that critical points of f_t are bounded (lemma 9).
- if we have points $M(t_0,c)$, not in $\mathcal{H}_\infty$, then there is infinitely many affine critical points for f_{t_0}, which is impossible since the singularities of f_{t_0} are isolated.

□

7. Examples

Example 1. Let $f_t = x(x^2y + tx + 1)$. Then $\mathcal{B}_{\mathit{aff}}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, $\lambda(t) = 1$ and $\deg f_t = 4$. The by theorem 4, f_0 and f_1 are topologically equivalent. These are examples of polynomials that are topologically but not algebraically equivalent, see [Bo].

Example 2. Let $f_t = (x+t)(xy+1)$. Then f_0 and f_1 are not topologically equivalent. One has $\mathcal{B}_\infty(t) = \varnothing$, $\mathcal{B}_{\mathit{aff}}(t) = \{0,t\}$ for $t \neq 0$, but $\mathcal{B}_\infty(0) = \{0\}$, $\mathcal{B}_{\mathit{aff}}(0) = \varnothing$. In fact the two affine critical points for f_t "escape at infinity" as t tends towards 0.

Example 3. Let $f_t = x\big(x(y + tx^2) + 1\big)$. Then f_0 is topologically equivalent to f_1. We have for all $t \in [0,1]$, $\mathcal{B}_{\mathit{aff}}(t) = \varnothing$, $\mathcal{B}_\infty(t) = \{0\}$, and $\lambda(t) = 1$, but $\deg f_t = 4$ for $t \neq 0$ while $\deg f_0 = 3$.

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