NEWTON POLYGONS AND FAMILIES OF POLYNOMIALS

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ABSTRACT. We consider a continuous family (f_s), $s \in [0,1]$ of complex polynomials in two variables with isolated singularities, that are Newton non-degenerate. We suppose that the Euler characteristic of a generic fiber is constant. We firstly prove that the set of critical values at infinity depends continuously on s, and secondly that the degree of the f_s is constant (up to an algebraic automorphism of $\mathbb{C}^2$).

1. INTRODUCTION

We consider a family $(f_s)_{s\in[0,1]}$ of complex polynomials in two variables with isolated singularities. We suppose that coefficients are continuous functions of s. For all s, there exists a finite *bifurcation set* $\mathcal{B}(s)$ such that the restriction of f_s above $\mathbb{C} \setminus \mathcal{B}(s)$ is a locally trivial fibration. It is known that $\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_\infty(s)$, where $\mathcal{B}_{\mathit{aff}}(s)$ is the set of *affine critical values*, that is to say the image by f_s of the critical points; $\mathcal{B}_\infty(s)$ is the set of *critical values at infinity*. For $c \notin \mathcal{B}(s)$, the Euler characteristic verifies $\chi(f_s^{-1}(c)) = \mu(s) + \lambda(s)$, where $\mu(s)$ is the *affine Milnor number* and $\lambda(s)$ is the *Milnor number at infinity*.

We will be interested in families such that the sum $\mu(s)+\lambda(s)$ is constant. These families are interesting in the view of μ-constant type theorem, see [HZ, HP, Ti, Bo, BT]. We say that a multi-valued function $s \mapsto F(s)$ is *continuous* if at each point $\sigma \in [0,1]$ and at each value $c(\sigma) \in F(\sigma)$ there is a neighborhood I of σ such that for all $s \in I$, there exists $c(s) \in F(s)$ near $c(\sigma)$. F is *closed*, if, for all points $\sigma \in [0,1]$, for all sequences $c(s) \in F(s)$, $s \neq \sigma$, such that $c(s) \to c(\sigma) \in \mathbb{C}$ as $s \to \sigma$, then $c(\sigma) \in F(\sigma)$. It it is well-known that $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is a continuous multi-valued function. But it is not necessarily closed: for example $f_s(x,y) = (x-s)(xy-1)$, then for $s \neq 0$, $\mathcal{B}_{\mathit{aff}}(s) = \{0, s\}$ but $\mathcal{B}_{\mathit{aff}}(0) = \varnothing$.

We will prove that $s \mapsto \mathcal{B}_\infty(s)$ and $s \mapsto \mathcal{B}(s)$ are closed continuous functions under some assumptions.

Theorem 1. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials such that $\mu(s)+\lambda(s)$ is constant and such that f_s is (Newton) non-degenerate for all $s \in [0,1]$, then the multi-valued function $s \mapsto \mathcal{B}_\infty(s)$ is continuous and closed.*

Remark. As a corollary we get the answer to a question of D. Siersma: is it possible to find a family with $\mu(s)+\lambda(s)$ constant such that $\lambda(0) > 0$ (equivalently $\mathcal{B}_\infty(0) \neq \varnothing$) and $\lambda(s) = 0$ (equivalently $\mathcal{B}_\infty = \varnothing$) for $s \in$

$]0,1]$? Theorem 1 says that it is not possible for non-degenerate polynomials. Moreover for a family with $\mu(s)+\lambda(s)$ constant and $\lambda(s)>0$ for $s\in]0,1]$ we have $\lambda(0)\geqslant\lambda(s)>0$ by the (lower) semi-continuity of $\mu(s)$. In the case of a $\mathcal{FISI}$ deformation of polynomials of constant degree with a non-singular total space, the answer can be deduced from [ST, Theorem 5.4].

Remark. Theorem 1 does not imply that $\mu(s)$ and $\lambda(s)$ are constant. For example let the family $f_s(x,y)=x^2y^2+sxy+x$. Then for $s=0$, $\mu(0)=0$, $\lambda(0)=2$ with $\mathcal{B}_\infty(0)=\{0\}$, and for $s\neq 0$, $\mu(s)=1$, $\lambda(s)=1$ with $\mathcal{B}_{\mathit{aff}}(s)=\{0\}$ and $\mathcal{B}_\infty(s)=\{-\frac{s^2}{4}\}$.

The multi-valued function $s\mapsto\mathcal{B}_{\mathit{aff}}(s)$ is continuous but not necessarily closed even if $\mu(s)+\lambda(s)$ is constant, for example (see [Ti]): $f_s(x,y)=x^4-x^2y^2+2xy+sx^2$, then $\mu(s)+\lambda(s)=5$. We have $\mathcal{B}_{\mathit{aff}}(0)=\{0\}$, $\mathcal{B}_\infty(0)=\{1\}$ and for $s\neq 0$, $\mathcal{B}_{\mathit{aff}}=\{0,1-\frac{s^2}{4}\}$, $\mathcal{B}_\infty(s)=\{1\}$. We notice that even if $s\mapsto\mathcal{B}_{\mathit{aff}}(s)$ is not closed, the map $s\mapsto\mathcal{B}(s)$ is closed. This is expressed in the following corollary (of Theorems 1 and 3):

Corollary 2. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials such that $\mu(s)+\lambda(s)$ is constant and such that f_s is non-degenerate for all $s\in[0,1]$. Then the multi-valued function $s\mapsto\mathcal{B}(s)$ is continuous and closed.*

We are now interested in the constancy of the degree; in all hypotheses of global μ-constant theorems the degree of the f_s is supposed not to change (see [HZ, HP, Bo, BT]) and it is the only non-topological hypothesis. We prove that for non-degenerate polynomials in two variables the degree is constant except for a few cases, where the family is of quasi-constant degree. We will define in a combinatoric way in paragraph 3 what a family of *quasi-constant degree* is, but the main point is to know that such a family is of constant degree up to some algebraic automorphism of $\mathbb{C}^2$. More precisely, for each value $\sigma\in[0,1]$ there exists $\Phi\in\operatorname{Aut}\mathbb{C}^2$ such $f_s\circ\Phi$ is of constant degree, for s in a neighborhood of σ. For example the family $f_s(x,y)=x+sy^2$ is of quasi-constant degree while the family $f_s(x,y)=sxy+x$ is not.

Theorem 3. *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials such that $\mu(s)+\lambda(s)$ is constant and such that f_s is non-degenerate for all $s\in]0,1]$, then either $(f_s)_{s\in[0,1]}$ is of constant degree or $(f_s)_{s\in[0,1]}$ is of quasi-constant degree.*

Remark. In theorem 3, f_0 may be degenerate.

As a corollary we get a μ-constant theorem without hypothesis on the degree:

Theorem 4. *Let $(f_s)_s\in[0,1]$ be a family of polynomials in two variables with isolated singularities such that the coefficients are continuous function of s. We suppose that f_s is non-degenerate for $s\in]0,1]$, and that the integers $\mu(s)+\lambda(s),\#\mathcal{B}(s)$ are constant ($s\in[0,1]$) then the polynomials f_0 and f_1 are topologically equivalent.*

It is just the application of the μ-constant theorem of [Bo], [BT] to the family (f_s) or $(f_s \circ \Phi)$. Two kinds of questions can be asked : are Theorems 1 and 3 true for degenerate polynomials? are they true for polynomials in more than 3 variables? I would like to thank Prof. Günter Ewald for discussions concerning Theorem 3 in n variables (that unfortunately only yield that the given proof cannot be easily generalized).

2. Tools

2.1. **Definitions.** We will recall some basic facts about Newton polygons, see [Ko], [CN], [NZ]. Let $f \in \mathbb{C}[x,y]$, $f(x,y) = \sum_{(p,q)\in\mathbb{N}^2} a_{p,q}x^py^q$. We denote $\operatorname{supp}(f) = \{(p,q) \mid a_{p,q} \neq 0\}$, by abuse $\operatorname{supp}(f)$ will also denote the set of monomials $\{x^py^q \mid (p,q) \in \operatorname{supp}(f)\}$. $\Gamma_-(f)$ is the convex closure of $\{(0,0)\} \cup \operatorname{supp}(f)$, $\Gamma(f)$ is the union of closed faces which do not contain $(0,0)$. For a face γ, $f_\gamma = \sum_{(p,q)\in\gamma} a_{p,q}x^py^q$. The polynomial f is *(Newton) non-degenerate* if for all faces γ of $\Gamma(f)$ the system

$$\frac{\partial f_\gamma}{\partial x}(x,y) = 0; \quad \frac{\partial f_\gamma}{\partial y}(x,y) = 0$$

has no solution in $\mathbb{C}^* \times \mathbb{C}^*$.

We denote by S the area of $\Gamma_-(f)$, by a the length of the intersection of $\Gamma_-(f)$ with the x-axis, and by b the length of the intersection of $\Gamma_-(f)$ with the y-axis (see Figure 1). We define

$$\nu(f) = 2S - a - b + 1.$$

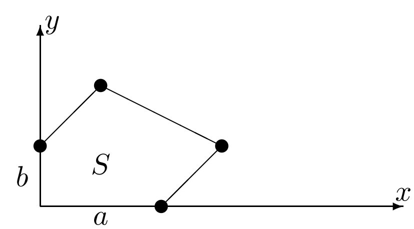

FIGURE 1. Newton polygon of f and $\nu(f) = 2S - a - b + 1$.

2.2. **Milnor numbers.** The following result is due to Pi. Cassou-Noguès [CN], it is an improvement of Kouchnirenko's result.

Theorem 5. *Let $f \in \mathbb{C}[x,y]$ with isolated singularities. Then*

(1) $\mu(f) + \lambda(f) \leqslant \nu(f)$.
(2) *If f is non-degenerate then $\mu(f) + \lambda(f) = \nu(f)$.*

2.3. **Critical values at infinity.** We recall the result of A. Néméthi and A. Zaharia on how to estimate critical values at infinity. A polynomial $f \in \mathbb{C}[x,y]$ is *convenient for the x-axis* if there exists a monomial x^a in $\operatorname{supp}(f)$ $(a>0)$; f is *convenient for the y-axis* if there exists a monomial y^b in $\operatorname{supp}(f)$ $(b>0)$; f is *convenient* if it is convenient for the x-axis and the y-axis. It is well-known (see [Br]) that:

Lemma 6. *A non-degenerate and convenient polynomial with isolated singularities has no critical value at infinity: $\mathcal{B}_\infty = \varnothing$.*

Let $f \in \mathbb{C}[x,y]$ be a polynomial with $f(0,0)=0$ not depending only on one variable. Let γ_x and γ_y the two faces of $\Gamma_-(f)$ that contain the origin. If f is convenient for the x-axis then we set $\mathcal{C}_x = \varnothing$ otherwise γ_x is not included in the x-axis and we set

$$\mathcal{C}_x = \left\{ f_{\gamma_x}(x,y) \mid (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \text{ and } \frac{\partial f_{\gamma_x}}{\partial x}(x,y) = \frac{\partial f_{\gamma_x}}{\partial y}(x,y) = 0 \right\}.$$

In a similar way we define $\mathcal{C}_y$.

A result of [NZ, Proposition 6] is:

Theorem 7. *Let $f \in \mathbb{C}[x,y]$ be a non-degenerate and non-convenient polynomial with $f(0,0)=0$, not depending only on one variable. The set of critical values at infinity of f is*

$$\mathcal{B}_\infty = \mathcal{C}_x \cup \mathcal{C}_y \quad \text{or} \quad \mathcal{B}_\infty = \{0\} \cup \mathcal{C}_x \cup \mathcal{C}_y.$$

Unfortunately this theorem does not determine whether $0 \in \mathcal{B}_\infty$ (and notice that the value 0 may be already included in $\mathcal{C}_x$ or $\mathcal{C}_y$). This value 0 is treated in the following lemma.

Lemma 8. *Let $f \in \mathbb{C}[x,y]$ be a non-degenerate and non-convenient polynomial, with isolated singularities and with $f(0,0)=0$. Then*

$$\mathcal{B}_\infty = \mathcal{B}_{\infty,x} \cup \mathcal{B}_{\infty,y}$$

where we define:

1. *if f is convenient for the x-axis then $\mathcal{B}_{\infty,x} := \varnothing$;*
2. *otherwise there exists $x^p y$ in* $\operatorname{supp}(f)$ *where $p \geqslant 0$ is supposed to be maximal;*
   - (a) *If $x^p y$ is in a face of $\Gamma_-(f)$ then $\mathcal{B}_{\infty,x} := \mathcal{C}_x$ and $0 \notin \mathcal{B}_{\infty,x}$;*
   - (b) *If $x^p y$ is not in a face of $\Gamma_-(f)$ then $\mathcal{B}_{\infty,x} := \{0\} \cup \mathcal{C}_x$;*
3. *we set a similar definition for $\mathcal{B}_{\infty,y}$.*

Theorem 7 and its refinement Lemma 8 enable to calculate $\mathcal{B}_\infty$ from $\operatorname{supp}(f)$. The different cases of Lemma 8 are pictured in Figures 2 and 3.

Proof. As f is non-convenient with $f(0,0)=0$ we may suppose that f is non-convenient for the x-axis so that $f(x,y) = yk(x,y)$. But f has isolated singularities, so y does not divide k. Then there is a monomial $x^p y \in \operatorname{supp}(f)$, we can suppose that $p \geqslant 0$ is maximal among monomials $x^k y \in \operatorname{supp}(f)$.

Let $d = \deg f$. Let $\bar{f}(x,y,z) - cz^d$ be the homogeneization of $f(x,y) - c$; at the point at infinity $P = (1:0:0)$, we define $g_c(y,z) = \bar{f}(1,y,z) - cz^d$. Notice that only $(1:0:0)$ and $(0:1:0)$ can be singularities at infinity for f. The value 0 is a critical value at infinity for the point at infinity P (that is to say $0 \in \mathcal{B}_{\infty,x}$) if and only if $\mu_P(g_0) > \mu_P(g_c)$ where c is a generic value.

The Newton polygon of the germ of singularity g_c can be computed from the Newton polygon $\Gamma(f)$, for $c \neq 0$, see [NZ, Lemma 7]. If A, B, O are the points on the Newton diagram of coordinates $(d,0), (0,d), (0,0)$, then the Newton diagram of g_c has origin A with y-axis AB, z-axis AO, and the convex closure of $\mathrm{supp}(g_c)$ corresponds to $\Gamma_-(f)$.

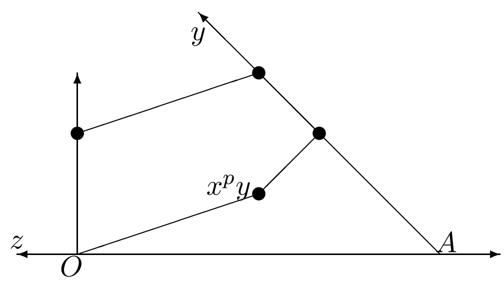

FIGURE 2. Newton polygon of g_c. First case: $0 \notin \mathcal{B}_{\infty,x}$.

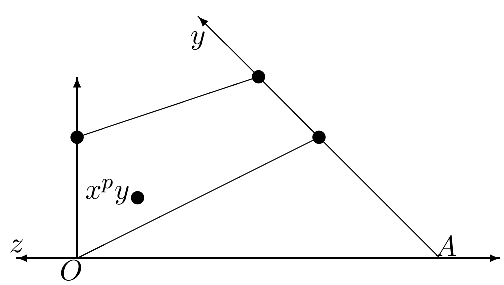

FIGURE 3. Newton polygon of g_c. Second case: $0 \in \mathcal{B}_{\infty,x}$.

We denote by Δ_c the Newton polygon of the germ g_c, for a generic value c, Δ_c is non-degenerate and $\mu_P(g_c) = \nu(\Delta_c)$. The Newton polygon Δ_0 has no common point with the z-axis AO but ν may be defined for non-convenient series, see [Ko, Definition 1.9].

If $x^p y$ is in the face γ_x of $\Gamma_-(f)$ then Δ_0 is non-degenerate and $\nu(\Delta_0) = \nu(\Delta_c)$, then by [Ko, Theorem 1.10] $\mu_P(g_0) = \nu(\Delta_0)$ and $\mu_P(g_c) = \nu(\Delta_c)$. So $\mu_P(g_0) = \mu_P(g_c)$ and 0 is not a critical value at infinity for the point P : $0 \notin \mathcal{B}_{\infty,x}$.

If $x^p y$ is not in a face of $\Gamma_-(f)$ then there is a triangle Δ_c that disappears in Δ_0, by the positivity of ν (see below) we have $\nu(\Delta_0) > \nu(\Delta_c)$, then by [Ko, Theorem 1.10]: $\mu_P(g_0) \geqslant \nu(\Delta_0) > \nu(\Delta_c) = \mu_P(g_c)$. So we have $0 \in \mathcal{B}_{\infty,x}$. □ □

2.4. **Additivity and positivity.** We need a variation of Kouchnirenko's number ν. Let T be a polytope whose vertices are in $\mathbb{N}\times\mathbb{N}$, $S>0$ the area of T, a the length of the intersection of T with the x-axis, and b the length of the intersection of T with the y-axis. We define

$$\tau(T) = 2S - a - b, \text{ so that, } \nu(T) = \tau(T) + 1.$$

It is clear that τ is additive: $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2) - \tau(T_1 \cap T_2)$, and in particular if $T_1 \cap T_2$ has null area then $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2)$. This formula enables us to argue on triangles only (after a triangulation of T).

Let T_0 be the triangle defined by the vertices $(0,0),(1,0),(0,1)$, we have $\nu(T_0) = -1$. We have the following facts, for every triangle $T \neq T_0$:

(1) $\nu(T) \geqslant 0$;
(2) $\nu(T) = 0$ if and only if T has an edge contained in the x-axis or the y-axis and the height of T (with respect to this edge) is 1.

Remark. The formula of additivity can be generalized in the n-dimensional case, but the positivity can not. Here is a counter-example found by Günter Ewald: Let $n=4$, a a positive integer and let T be the polytope whose vertices are: $(1,0,0,0)$, $(1+a,0,0,0)$, $(1,1,1,0)$, $(1,2,1,0)$, $(1,1,1,1)$ then $\tau(T) = \nu(T) + 1 = -a < 0$.

2.5. **Families of polytopes.** We consider a family $(f_s)_{s\in[0,1]}$ of complex polynomials in two variables with isolated singularities. We suppose that $\mu(s)+\lambda(s)$ remains constant. We denote by $\Gamma(s)$ the Newton polygon of f_s. We suppose that f_s is non-degenerate for $s \in]0,1]$.

We will always assume that the only critical parameter is $s=0$. We will say that a monomial x^py^q *disappears* if $(p,q) \in \operatorname{supp}(f_s) \setminus \operatorname{supp}(f_0)$ for $s \neq 0$. By extension a triangle of $\mathbb{N}\times\mathbb{N}$ disappears if one of its vertices (which is a vertex of $\Gamma(s)$, $s \neq 0$) disappears. Now after a triangulation of $\Gamma(s)$ we have a finite number of triangles T that disappear (see Figure 4, on pictures of the Newton diagram, a plain circle is drawn for a monomial that does not disappear and an empty circle for monomials that disappear).

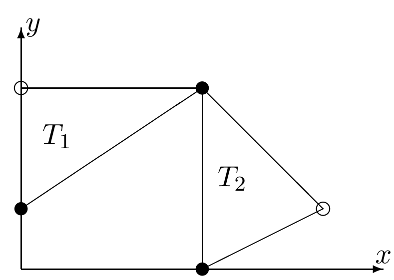

FIGURE 4. Triangles that disappear.

We will widely use the following result, under the hypotheses of Theorem 3:

Lemma 9. *Let $T \neq T_0$ be a triangle that disappears then $\tau(T) = 0$.*

Proof. We suppose that $\tau(T) > 0$. By the additivity and positivity of τ we have for $s \in]0,1]$:

$$\nu(s) = \nu(\Gamma(s)) \geqslant \nu(\Gamma(0)) + \tau(T) > \nu(0).$$

Then by Theorem 5,

$$\mu(s) + \lambda(s) = \nu(s) > \nu(0) \geqslant \mu(0) + \lambda(0).$$

This gives a contradiction with $\mu(s) + \lambda(s) = \mu(0) + \lambda(0)$.

We remark that we do not need f_0 to be non-degenerate because in all cases we have $\nu(0) \geqslant \mu(0) + \lambda(0)$. □ □

3. Constancy of the degree

3.1. **Families of quasi-constant degree.** Let $\sigma \in [0,1]$, we choose a small enough neighborhood I of σ. Let $\mathcal{M}_\sigma$ be the set of monomials that disappear at σ: $\mathcal{M}_\sigma = \operatorname{supp}(f_s) \setminus \operatorname{supp}(f_\sigma)$ for $s \in I \setminus \{\sigma\}$. The family $(f_s)_{s\in[0,1]}$ is of *quasi-constant degree at σ* if

there exists $x^p y^q \in \operatorname{supp}(f_\sigma)$ such that

$$(\forall x^{p'}y^{q'} \in \mathcal{M}_\sigma \quad (p > p') \text{ or } (p = p' \text{ and } q > q'))$$
$$\text{or } (\forall x^{p'}y^{q'} \in \mathcal{M}_\sigma \quad (q > q') \text{ or } (q = q' \text{ and } p > p')).$$

The family $(f_s)_{s\in[0,1]}$ is of *quasi-constant degree* if it is of quasi-constant degree at each point σ of $[0,1]$. The terminology is justified by the following remark:

Lemma 10. *If (f_s) is of quasi-constant degree at $\sigma \in [0,1]$, then there exists $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such that $\deg f_s \circ \Phi$ is constant in a neighborhood of σ.*

The proof is simple: suppose that $x^p y^q$ is a monomial of $\operatorname{supp}(f_\sigma)$ such that for all $x^{p'}y^{q'} \in \mathcal{M}_\sigma$, $p > p'$ or ($p = p'$ and $q > q'$). We set $\Phi(x,y) = (x + y^\ell, y)$ with $\ell \gg 1$. Then the monomial of highest degree in $f_s \circ \Phi$ is $y^{q+p\ell}$ and does not disappear at σ. For example let $f_s(x,y) = xy + sy^3$, we set $\Phi(x,y) = (x + y^3, y)$ then $f_s \circ \Phi(x,y) = y^4 + xy + sy^3$ is of constant degree.

We prove Theorem 3. We suppose that the degree changes, more precisely we suppose that $\deg f_s$ is constant for $s \in]0,1]$ and that $\deg f_0 < \deg f_s$, $s \in]0,1]$. As the degree changes the Newton polygon $\Gamma(s)$ cannot be constant, that means that at least one vertex of $\Gamma(s)$ disappears.

3.2. **Exceptional case.** We suppose that f_0 is a one-variable polynomial, for example $f_0 \in \mathbb{C}[y]$. As f_0 has isolated singularities then $f_0(x,y) = a_0 y + b_0$, so $\mu(0) = \lambda(0) = 0$, then for all s, $\mu(s) = \lambda(s) = 0$. So $\nu(s) = \nu(\Gamma(s)) = 0$, then $\deg_y f_s = 1$, and $f_s(x,y) = a_s y + b_s(x)$, so $(f_s)_{s\in[0,1]}$ is a family of quasi-constant degree (see Figure 5). We exclude this case for the end of the proof.

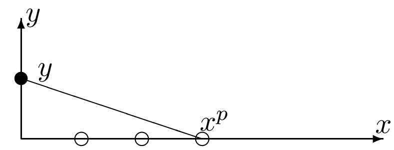

FIGURE 5. Case $f_0 \in \mathbb{C}[y]$.

3.3. **Case to exclude.** We suppose that a vertex $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears. Then there exists a triangle T that disappears whose faces are not contained in the axis. Then $\tau(T) > 0$ that contradicts Lemma 9 (see Figure 6).

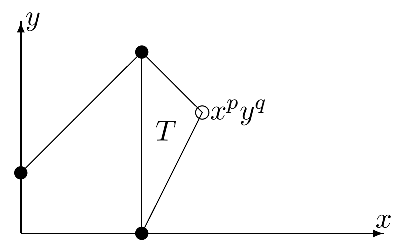

FIGURE 6. Case where a monomial $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears.

3.4. **Case where a monomial x^a or y^b disappears (but not both).** If, for example the monomial y^b of $\Gamma(s)$ disappears and x^a does not, then we choose a monomial $x^p y^q$, with maximal p, among monomials in $\mathrm{supp}(f_s)$. Certainly $p \geqslant a > 0$. We also suppose that q is maximal among monomials $x^p y^k \in \mathrm{supp}(f_s)$. If $q = 0$ then $p = a$, and the monomial $x^p y^q = x^a$ does not disappear (by assumption). If $q > 0$ then $x^p y^q$ cannot disappear (see above). In both cases the monomial $x^p y^q$ proves that (f_s) is of quasi-constant degree.

3.5. **Case where both x^a and y^b disappear.**
Sub-case : No monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$. Then there is an area T with $\tau(T) > 0$ that disappears (see Figure 7). Contradiction.

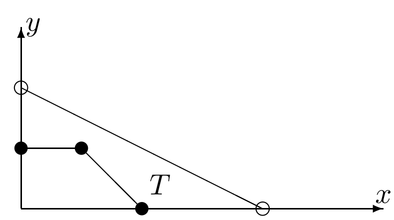

FIGURE 7. Sub-case : no monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$.

Sub-case : there exists a monomial x^py^q in $\Gamma(s)$, $p > 0, q > 0$. We know that x^py^q is in $\Gamma(0)$ because it cannot disappear. As $\deg f_0 < \deg f_s$, a monomial x^py^q that does not disappear verifies $\deg x^py^q = p+q < \deg f_s$, ($s \in]0,1]$). So the monomial of highest degree is x^a or y^b. We will suppose that it is y^b, so $d = b$, and the monomial y^b disappears. Let x^py^q be a monomial of $\Gamma(s)$, $p, q > 0$ with minimal q. By assumption such a monomial exists. Then certainly we have $q = 1$, otherwise there exists a region T that disappears with $\tau(T) > 0$ (on Figure 8 the regions T_1 and T_2 verify $\nu(T_1) = 0$ and $\nu(T_2) = 0$). For the same reason the monomial $x^{p'}y^{q'}$ with minimal p' verifies $p' = 1$.

We look at the segments of $\Gamma(s)$, starting from $y^b = y^d$ and ending at x^a. The first segment is from y^d to $xy^{q'}$, ($p' = 1$) and we know that $p'+q' < d$ so the slope of this segment is strictly less than -1. By the convexity of $\Gamma(s)$ all the following slopes are strictly less than -1. The last segment is from x^py to x^a, with a slope strictly less than -1, so $a \leqslant p$. Then the monomial x^py gives that $(f_s)_{s\in[0,1]}$ is of quasi-constant degree.

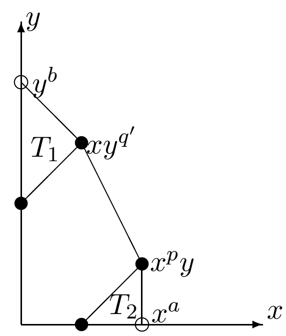

FIGURE 8. Sub-case : existence of monomials x^py^q in $\Gamma(s)$, $p > 0, q > 0$.

4. Continuity of the critical values

We now prove Theorem 1. We will suppose that $s = 0$ is the only problematic parameter. In particular $\Gamma(s)$ is constant for all $s \in]0,1]$.

4.1. **The Newton polygon changes.** That is to say $\Gamma(0) \neq \Gamma(s)$, $s \neq 0$. As in the proof of Theorem 3 (see paragraph 3) we remark:

- If f_0 is a one-variable polynomial then $\mathcal{B}_\infty(s) = \varnothing$ for all $s \in [0,1]$.
- A vertex x^py^q, $p > 0, q > 0$ of $\Gamma(s)$ cannot disappear.

So we suppose that a monomial x^a of $\Gamma(s)$ disappears (a similar proof holds for y^b). Then for $s \in]0,1]$ the monomial x^a is in $\Gamma(s)$, so there are no critical values at infinity for f_s at the point $P = (1:0:0)$. If $\Gamma(0)$ contains a monomial $x^{a'}$, $a' > 0$ then there are no critical values at infinity for f_0 at the point P . So we suppose that all monomials x^k disappear.

Then a monomial $x^p y^q$ of $\operatorname{supp}(f_0)$ with minimal $q > 0$, verifies $q = 1$, otherwise there would exist a region T with $\tau(T) > 0$ (in contradiction with the constancy of $\mu(s) + \lambda(s)$, see Lemma 9). And for the same reason if we choose $x^p y$ in $\operatorname{supp}(f_0)$ with maximal p then $p > 0$ and $x^p y \in \Gamma(0)$. Now the edge of $\Gamma_-(f_0)$ that contains the origin and the monomial $x^p y$ (with maximal p) begins at the origins and ends at $x^p y$ (so in particular there is no monomial $x^{2p}y^2$, $x^{3p}y^3$ in $\operatorname{supp}(f_0)$). Now from Theorem 7 and Lemma 8 we get that there are no critical values at infinity for f_0 at P.

So in case where $\Gamma(s)$ changes, we have for all $s \in [0,1]$, $\mathcal{B}_\infty(s) = \varnothing$.

4.2. **The Newton polygon is constant : case of non-zero critical values.** We now prove the following lemma that ends the proof of Theorem 1.

Lemma 11. *Let a family $(f_s)_{s\in[0,1]}$ such that f_s is non-degenerate for all $s \in [0,1]$ and $\Gamma(s)$ is constant, then the multi-valued function $s \mapsto \mathcal{B}_\infty(s)$ is continuous and closed.*

In this paragraph and the next one we suppose that $f_s(0,0) = 0$, that is to say the constant term of f_s is zero. We suppose that $c(0) \in \mathcal{B}_\infty(0)$ and that $c(0) \neq 0$. Then $c(0)$ has been obtained by the result of Néméthi-Zaharia (see Theorem 7). There is a face γ of $\Gamma_-(f_0)$ that contains the origin such that $c(0)$ is in the set:

$$\mathcal{C}_\gamma(0) = \left\{ (f_0)_\gamma(x,y) \mid (x,y) \in (\mathbb{C}^*)^2 \text{ and } \frac{\partial (f_0)_\gamma}{\partial x}(x,y) = \frac{\partial (f_0)_\gamma}{\partial y}(x,y) = 0 \right\}.$$

Now, as $\Gamma(s)$ is constant, γ is a face of $\Gamma_-(s)$ for all s. There exists a family of polynomials $h_s \in \mathbb{C}[t]$ and a monomial $x^p y^q$ ($p,q > 0$, $\gcd(p,q) = 1$) such that $(f_s)_\gamma(x,y) = h_s(x^p y^q)$. The family (h_s) is continuous (in s) and is of constant degree (because $\Gamma(s)$ is constant). The set $\mathcal{C}_\gamma(0)$ and more generally the set $\mathcal{C}_\gamma(s)$ can be computed by

$$\mathcal{C}_\gamma(s) = \left\{ h_s(t) \mid t \in \mathbb{C}^* \text{ and } h_s'(t) = 0 \right\}.$$

As $c(0) \in \mathcal{C}_\gamma(0)$ there exists a $t_0 \in \mathbb{C}^*$ with $h_0'(t_0) = 0$, and for s near 0 there is a $t_s \in \mathbb{C}^*$ near t_0 with $h_s'(t_s) = 0$ (because $h_s'(t)$ is a continuous function of s of constant degree in t). Then $c(s) = h_s(t_s)$ is a critical value at infinity near $c(0)$ and we get the continuity.

4.3. **The Newton polygon is constant : case of the value 0.** We suppose that $c(0) = 0 \in \mathcal{B}_\infty(0)$ and that $f_s(x,y) = yk_s(x,y)$. We will deal with the point at infinity $P = (1:0:0)$, the point $(0:1:0)$ is treated in a similar way. Let $x^p y$ be a monomial of $\operatorname{supp}(f_s)$ with maximal $p \geqslant 0$, $s \neq 0$. If $x^p y$ is not in a face of $\Gamma(s)$ then $0 \in \mathcal{B}_\infty(s)$ for all $s \in [0,1]$, and we get the continuity. Now we suppose that $x^p y$ is in a face of $\Gamma(s)$; then $x^p y$ disappears otherwise 0 is not a critical value at infinity (at the point P) for all $s \in [0,1]$. As $\Gamma(s)$ is constant then the face γ that contains the origin and $x^p y$ for $s \neq 0$ is also a face of $\Gamma(0)$, then there exists a monomial $(x^p y)^k$, $k > 1$ in $\operatorname{supp}(f_0)$. Then $(f_s)_\gamma = h_s(x^p y)$, $h_s \in \mathbb{C}[t]$. We have $\deg h_s > 1$, with $h_s(0) = 0$ (because $f(0,0) = 0$) and $h_0'(0) = 0$ (because $x^p y$ disappears). Then $0 \in \mathcal{C}_\gamma(0) \subset \mathcal{B}_\infty(0)$ but by continuity of h_s we have a critical value $c(s) \in \mathcal{C}_\gamma(s) \subset \mathcal{B}_\infty(s)$ such that $c(s)$ tends towards 0 (as $s \to 0$). It should be noticed that for $s \neq 0$, $c(s) \neq 0$.

In all cases we get the continuity of $\mathcal{B}_\infty(s)$.

4.4. **Proof of the closeness of $s \mapsto \mathcal{B}_\infty(s)$.** We suppose that $c(s) \in \mathcal{B}_\infty(s)$, is a continuous function of $s \neq 0$, with a limit $c(0) \in \mathbb{C}$ at $s = 0$. We have to prove that $c(0) \in \mathcal{B}_\infty(0)$. As there are critical values at infinity we suppose that $\Gamma(s)$ is constant.
Case $c(0) \neq 0$. Then for s near 0, $c(s) \neq 0$ by continuity, then $c(s)$ is obtained as a critical value of $h_s(t)$. By continuity $c(0)$ is a critical value of $h_0(t)$: $h_0(t_0) = c(0)$, $h_0'(t_0) = 0$; as $c(0) \neq 0$, $t_0 \neq 0$ (because $h_0(0) = 0$). Then $c(0) \in \mathcal{B}_\infty(0)$.
Case $c(0) = 0$. Then let $x^p y$ be the monomial of $\operatorname{supp}(f_s)$, $s \neq 0$, with maximal p. By Lemma 8 if $x^p y \notin \Gamma(s)$ for $s \in]0,1]$ then $0 \in \mathcal{B}_\infty(s)$ for all $s \in [0,1]$ and we get closeness. If $x^p y \in \Gamma(s)$, $s \neq 0$, then as $c(s) \to 0$ we have that $x^p y$ disappears, so $x^p y \notin \Gamma(0)$, then by Lemma 8, $c(0) = 0 \in \mathcal{B}_\infty(0)$.

4.5. **Proof of the closeness of $s \mapsto \mathcal{B}(s)$.** We now prove Corollary 2. The multi-valued function $s \mapsto \mathcal{B}(s)$ is continuous because $\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_\infty(s)$ and $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$, $s \mapsto \mathcal{B}_\infty(s)$ are continuous. For closeness, it remains to prove that if $c(s) \in \mathcal{B}_{\mathit{aff}}(s)$ is a continuous function with a limit $c(0) \in \mathbb{C}$ at $s = 0$ then $c(0) \in \mathcal{B}(0)$.

We suppose that $c(0) \notin \mathcal{B}_{\mathit{aff}}(0)$. There exist critical points $Q_s = (x_s, y_s) \in \mathbb{C}^2$ of f_s with $f_s(x_s, y_s) = c(s)$, $s \neq 0$. We can extract a countable set $\mathcal{S}$ of $]0,1]$ such that the sequence $(Q_s)_{s \in \mathcal{S}}$ converges towards P in $\mathbb{C}P^2$. As $c(0) \notin \mathcal{B}_{\mathit{aff}}(0)$ we have that P relies on the line at infinity and we may suppose that $P = (0:1:0)$.

By Theorem 3 we may suppose, after an algebraic automorphism of $\mathbb{C}^2$ if necessary, that $d = \deg f_s$ is constant. Now we look at $g_{s,c}(x,z) = \bar{f}_s(x,1,z) - cz^d$. The critical point Q_s of f_s with critical value $c(s)$ gives a critical point $Q_s' = (\frac{x_s}{y_s}, \frac{1}{y_s})$ of $g_{s,c(s)}$ with critical value 0 (see [Bo, Lemma 21]). Then by semi-continuity of the local Milnor number on the fiber $g_{s,c(s)}^{-1}(0)$ we have $\mu_P(g_{0,c(0)}) \geqslant \mu_P(g_{s,c(s)}) + \mu_{Q_s'}(g_{s,c(s)}) > \mu_P(g_{s,c(s)})$. As $\mu(s) + \lambda(s)$ is constant we have $\mu_P(g_{s,c})$ constant for a generic c (see [ST, Corollary 5.2] or [BT]). Then we have $\mu_P(g_{0,c(0)}) - \mu_P(g_{0,c}) > \mu_P(g_{s,c(s)}) - \mu_P(g_{s,c}) \geqslant 0$. Then $c(0) \in \mathcal{B}_\infty(0)$. And we get closeness for $s \mapsto \mathcal{B}(s)$.

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