COMPUTATION OF MILNOR NUMBERS AND CRITICAL VALUES AT INFINITY

ARNAUD BODIN

ABSTRACT. We describe how to compute topological objects associated to a complex polynomial map of $n \geqslant 2$ variables with isolated singularities. These objects are: the affine critical values, the affine Milnor numbers for all irregular fibers, the critical values at infinity, and the Milnor numbers at infinity for all irregular fibers. Then for a family of polynomials we detect parameters where the topology of the polynomials can change. Implementation and examples are given with the computer algebra system SINGULAR.

1. INTRODUCTION

1.1. **Review on the local case.** Let $g : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ be a germ of polynomial map with isolated singularities. One of the most important topological object attached to g is its *local Milnor number* [Mi]:

$$\mu_0 = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/\mathrm{Jac}(g)$$

where $\mathrm{Jac}(g) = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n})$ is the Jacobian ideal of g. It is possible to compute μ_0 with the help of a Gröbner base. For example such a computation motivates the computer algebra system SINGULAR, [GPS].

Now we consider a family $(g_s)_{s\in[0,1]}$, with $g_s : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ germs of isolated singularities, such that g_s is a smooth function of s. To each $s \in [0,1]$ we associate the local Milnor number $\mu_0(g_s)$. The main topological result for families is Lê-Ramanujam-Timourian μ-constant theorem.

Theorem 1 ([LR, Ti]). *If $n \neq 3$ and $\mu_0(g_s)$ is constant ($s \in [0,1]$) then the family $(g_s)_{s\in[0,1]}$ is a topologically trivial family.*

1.2. **Motivation and aims for the global case.** Now we consider a polynomial function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. The study of the topology of f is not just the glueing of local studies because of the behaviour of f at infinity, see [Br]. To the polynomial f we attach "Milnor numbers" μ, λ and finite sets of critical values $\mathcal{B}_{\mathit{aff}}$, $\mathcal{B}_\infty$, $\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty$ (see the definitions below). The first aim of this work is to compute these objects and to give the topology of the fibers $f^{-1}(c)$ for all $c \in \mathbb{C}$.

There is a global version of the local μ-constant theorem (see Theorem 2) where the Milnor number μ_0 is replaced by a *Milnor multi-integer* $\mathfrak{m} = (\mu, \#\mathcal{B}_{\mathit{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$.

Date: June 25, 2003.

In order to verify if $\mathfrak{m}(f_s)$ remains constant in a family $(f_s)_{s\in[0,1]}$ it is not possible to compute $\mathfrak{m}(f_s)$ for infinitely many values. The second aim of the work is to give (and compute) a finite set $\mathcal{S}'$ such that $\mathfrak{m}(f_s)$ is constant for $s \in [0,1] \setminus \mathcal{S}'$.

The rest of this section is devoted to the definitions and the results.

1.3. **Critical values.** Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geqslant 2$. By a result of Thom [Th] there is a minimal *set of critical values* $\mathcal{B}$ of point of $\mathbb{C}$ such that $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a fibration.

1.4. **Affine singularities.** We suppose that *affine singularities are isolated i.e.* that the set $\{x \in \mathbb{C}^n \mid \operatorname{grad}_f x = 0\}$ is a finite set. Let μ_c be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathit{aff}} = \big\{c \mid \mu_c > 0\big\} \quad \text{ and } \quad \mu = \sum_{c\in\mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*.

1.5. **Singularities at infinity.** See [Br]. Let d be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, let $f = f^d + f^{d-1} + \cdots + f^0$ where f^j is homogeneous of degree j. Let $\bar{f}(x,z)$ (with $x = (x_1, \ldots, x_n)$) be the homogenisation of f with the new variable z: $\bar{f}(x,z) = f^d(x) + f^{d-1}(x)z + \ldots + f^0(x)z^d$. Let

$$X = \big\{((x:z),t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x,z) - cz^d = 0\big\}.$$

Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(z = 0)$. The singular locus of X has the form $\Sigma \times \mathbb{C}$ where

$$\Sigma = \left\{(x:0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0\right\} \subset \mathcal{H}_\infty.$$

We suppose that f has *isolated singularities at infinity* that is to say that Σ is finite. This is always true for $n = 2$. We say that f has *strong isolated singularities at infinity* if

$$\Sigma' = \left\{(x:0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = 0\right\}$$

is finite.

For a point $(x:0) \in \mathcal{H}_\infty$, assume, for example, that $x = (x_1, \ldots, x_{n-1}, 1)$ and set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, z) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - cz^d.$$

Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of F_c at the point $(\check{x}, 0)$. If $(x:0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic s, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many c, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Let

$$\mathcal{B}_\infty = \big\{c \in \mathbb{C} \mid \lambda_c > 0\big\} \quad \text{ and } \quad \lambda = \sum_{c\in\mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity.*
We can now describe the set of critical values $\mathcal{B}$ as follows (see [HL] and [Pa]):

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

Moreover by [HL] and [ST] for all $c \in \mathbb{C}$, $f^{-1}(c)$ has the homotopy type of a wedge of $\mu + \lambda - \mu_c - \lambda_c$ spheres of real dimension $n-1$.

1.6. **Families of polynomials.** To a polynomial we associate its *Milnor multi-integer* $\mathfrak{m} = (\mu, \#\mathcal{B}_{\mathit{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$. Two polynomials maps $f, g : \mathbb{C}^n \longrightarrow \mathbb{C}$ are *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f \circ \Phi = \Psi \circ g$. The Milnor multi-integer is a topological invariant, that is to say if f and g are topologically equivalent then $\mathfrak{m}(f) = \mathfrak{m}(g)$. We recall a result of [Bo, BT] that is kind of converse of this property.
Let $(f_s)_{s\in[0,1]}$ be a family of polynomials, such that f_s has *strong* isolated singularities at infinity and isolated affine singularities for all $s \in [0,1]$. For each $s \in [0,1]$ we consider the Milnor multi-integer of f_s, $\mathfrak{m}(f_s) = (\mu(s), \#\mathcal{B}_{\mathit{aff}}(s), \lambda(s), \#\mathcal{B}_\infty(s), \#\mathcal{B}(s))$. We suppose that the coefficients of the family are polynomials in s and that the degree $\deg f_s$ is constant.

Theorem 2 ([Bo, BT])**.** *Let $n \neq 3$. If $\mathfrak{m}(f_s)$ is constant ($s \in [0,1]$), then f_0 is topologically equivalent to f_1.*

How to verify the hypotheses from a computable point of view ? It is not possible to compute $\mathfrak{m}(f_s)$ for infinitely many $s \in [0,1]$. But in fact $\mathfrak{m}(f_s)$ is constant except for finitely many s, we denote by $\mathcal{S}$ the set of these *critical parameters.*
In paragraph 4 we give a computation of a finite set $\mathcal{S}'$ such that

$$\mathcal{S} \subset \mathcal{S}'.$$

Now to check if a value $s \in \mathcal{S}'$ is in $\mathcal{S}$ we compute $\mathfrak{m}(f_s)$ and we compare it with $\mathfrak{m}(f_{s'})$ where s' is any value of $[0,1] \setminus \mathcal{S}'$; now $s \in \mathcal{S}$ if and only if $\mathfrak{m}(f_s) \neq \mathfrak{m}(f_{s'})$.

1.7. **Implementation.** The results of this paper have been implemented in two libraries `critic` and `defpol`. The first one enables to calculate all the objects defined above: $\mathcal{B}_{\mathit{aff}}$, μ, μ_c for $c \in \mathcal{B}_{\mathit{aff}}$; $\mathcal{B}_\infty$, λ, λ_c for $c \in \mathcal{B}_\infty$. These programs are written for Singular, [GPS]. It is based on polar curves and on the article of D. Siersma and M. Tibăr, [ST]. For polynomials in two variables ($n = 2$) a program in Maple has been written by G. Bailly-Maître, [BM], based on a discriminant formula of Hà H.V., [Ha]. For families of polynomials the second library computes a finite set $\mathcal{S}'$ that contains the critical parameters.
This research has partially been supported by a Marie Curie Individual Fellowship of the European Community (HPMF-CT-2001-01246).

2. Milnor numbers and critical values in affine space

2.1. **Milnor number.** The computation of the affine Milnor number μ is easy and well-known (see [GPS] for example). Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let J be the Jacobian ideal of the partial derivative $(\partial f/\partial x_i)_i$. Then μ is the vector space dimension (over $\mathbb{C}$) of a Gröbner basis of the quotient $\mathbb{C}[x_1, \ldots, x_n]/J$.

2.2. **Critical values.** We add a new variable t. We consider the variety

$$C = \left\{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid f(x) - t = 0 \ \text{ and } \ \operatorname{grad}_f x = 0\right\}.$$

The critical values are the projection of C on the t-coordinate: $\mathcal{B}_{\mathit{aff}} = \operatorname{pr}_t(C)$.

2.3. **Milnor number of a fiber.** Set $c \in \mathbb{C}$. We would like to compute μ_c the sum of the Milnor numbers of the points of $f^{-1}(c)$. Let J be the Jacobian ideal of f and set x a critical point. We denote by J_x the localization of J at x. Let $I_x = (t-c, J_x)$, the dimension of I_x is equal to the Milnor number of f at x. For $k \geqslant 1$ we consider $K_x^k = ((f-t)^k, I_x)$. Then $f(x) = c$ if and only if K_x has non-zero dimension (as a vector space). Moreover if $f(x) = c$ then, by the Nullstellensatz, $(f-t)^k$ is in I_x for a sufficiently large k. For such a k, the dimension of K_x is the Milnor number at x if $f(x) = c$, and it is 0 otherwise. Such a k is less or equal to the Milnor number at x, but k can often be chosen much less. The minimal k is the first integer such that the vector space dimension of K_x^k is equal to the one of K_x^{k+1}.

3. Milnor numbers and critical values at infinity

We give the computation of the objects at infinity and its implementation in Singular. We will suppose that f has isolated singularities at infinity, in fact computations are valid for a larger class of polynomials but it cannot be computed if f belongs to this class. The algorithm is based on the article of D. Siersma and M. Tibăr, [ST], that gives critical values at infinity and Milnor numbers at infinity with the help of polar curves.

3.1. **Working space.** We will work in $\mathbb{P}^n \times \mathbb{C}$, with the homogeneous coordinates of $\mathbb{P}^n$: $(x_1 : \ldots : x_n : z)$; we still need t which is a parameter or a variable depending on the context.
We recall that

$$X = \left\{((x:z),t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x,z) - tz^d = 0\right\}.$$

The part at infinity of X is $X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C})$:

$$X_\infty = \left\{((x:0),t) \in \mathbb{P}^n \times \mathbb{C} \mid f^d(x) = 0\right\},$$

Where $f = f^d + f^{d-1} + \cdots$ is the decomposition in homogeneous polynomials.
In Singular, we write:

```
ring r = 0, (x(1..n),z,t), dp;
poly f = ...;
poly fH = homog(f,z)-t*z^deg(f);
ideal X = fH;
ideal Xinf = z, fH;
```

3.2. **Polar curve.** Let k be in $\{1, \dots, n\}$. The polar curve $\mathcal{P}$ is the critical locus of the map $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}^2$ defined for $x = (x_1, \dots, x_n)$ by $\phi(x) = (f(x), x_k)$:

$$\mathcal{P} = \left\{ x \in \mathbb{C}^n \mid \frac{\partial f}{\partial x_i}(x) = 0, \forall i \neq k \right\}.$$

We have that $\mathcal{P}$ is a curve or is void. We call $\mathcal{P}_H$ the projective closure of $\mathcal{P}$. This curve intersects the hyperplane at infinity $\mathcal{H}_\infty$ in finitely many points.

```
ideal P = diff(f,x(1)),..., diff(f,x(k-1)), diff(f,x(k+1)),...;
ideal PH = homog(P,z);
```

The former objects can be viewed in X, we will also denote by $\mathcal{P}_H$, the set $(\mathcal{P}_H \times \mathbb{C}) \cap X$. In the chart $x_k = 1$, we denote the curve $\mathcal{P}_H$ by $\bar{\mathcal{C}}$. The "real" polar curve $\mathcal{C}$ in this chart is the closure of $\bar{\mathcal{C}} \setminus X_\infty$:

```
ideal Cbar = x(k)-1, PH, X;
ideal C = sat(Cbar,Xinf)[1];
```

3.3. **Critical values at infinity.** We need the following result of [ST]. A value c is a critical values at infinity if and only there is coordinate x_k and a point $(x : 0, t)$ in X_∞ (with $x_k \neq 0$) such that $(x : 0, t) \in \mathcal{C}$. That is to say $\mathcal{B}_\infty$ is the projection of $\mathcal{C}_\infty = X_\infty \cap \mathcal{C}$ on the space of parameters $t \in \mathbb{C}$.
Then the critical values are computed with:

```
ideal Cinf = z, C;
poly Binf = eliminate(Cinf,x(1)x(2)..x(n)z)[1];
```

The set of critical values at infinity are the roots of the polynomial `Binf`, which belongs to $\mathbb{C}[t]$.

3.4. **Milnor numbers at infinity.** Actually the results in [ST] are more precise. For a fixed t, let $X_t = \{(x : z, t) \in X\}$, this is a projective model for the fiber $f^{-1}(t)$.

Theorem 3 ([ST]). *The Milnor number at infinity at a point $(x : 0, t) \in \mathcal{C}_\infty$ is given by the intersection number (in X) of $\mathcal{C}$ with X_t at $(x : 0, t)$.*

So, for $c \in \mathcal{B}_\infty$, the Milnor number at infinity λ_c (for the chart $x_k \neq 0$), is equal to the sum of all intersection numbers of X_c and $\mathcal{C}$ in X_∞.
We compute an ideal I which correspond to $X_c \cap \mathcal{C}$, then we only deals with points at infinity by intersecting it this set with $z^q = 0$, for a sufficiently large q.

```
number c = ...;
ideal Xc = t-c, X;
ideal I = Xc, C;
ideal K = z^q, I;    // q >> 1
lambdac = vdim(std(K));
```

Once we have computed λ_c for all $c \in \mathcal{B}_\infty$, we have $\lambda = \sum_{c\in\mathcal{B}_\infty} \lambda_c$.

4. Families of polynomials

Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials in n variables. We suppose that the coefficients are polynomial functions of s and that for all $s \in [0,1]$, f_s has affine isolated singularities and strong isolated singularities at infinity. The implementation is similar to the one of paragraph 3 and will be omitted.

4.1. **Change in affine space.** It is not possible to compute infinitely many $\mu(s)$, so we have to detect a change of $\mu(s)$. The Milnor numbers $\mu(s)$ changes if and only if some critical points escape at infinity. Then we can detect critical parameters for μ as follows: Let $J = \left\{(x_1,\ldots,x_n,s) \in \mathbb{C}^n \times \mathbb{C} \mid \frac{\partial f_s}{\partial x_1} = \ldots, \frac{\partial f_s}{\partial x_n} = 0\right\}$ be the set of critical points (that corresponds to the Jacobian ideal in $\mathbb{C}[x_1,\ldots,x_n,s]$). Let $\bar{J}$ be the homogeneization of J with the new variable z, while s is considered as a parameter. The part at infinity of J corresponds to the ideal $J_\infty = \bar{J} \cap (z=0)$, and the affine part of J is $J_{\mathit{aff}} = \overline{\bar{J} \setminus J_\infty}$. Now the critical parameters for μ is $\mathrm{pr}_s(J_{\mathit{aff}}) \subset \mathbb{C}$, where pr_s is the projection to the s-coordinate.
It is possible to compute $\mathcal{B}_{\mathit{aff}}(s)$ for all $s \in [0,1]$ by a direct extension of the work of paragraph 2. Then we can compute the parameters where the cardinal of this set changes.

4.2. **Change at infinity.** Again it is not possible to compute infinitely many $\lambda(s)$. We extend the definition of paragraph 3 by adding a parameter s. We set $d = \deg f_s$ and

$$X = \left\{((x:z),t,s) \in \mathbb{P}^n \times \mathbb{C} \times \mathbb{C} \mid \bar{f}_s(x,z) - tz^d = 0\right\}.$$

The part at infinity of X is $X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C} \times \mathbb{C})$:

$$X_\infty = \left\{((x:0),t,s) \in \mathbb{P}^n \times \mathbb{C} \mid f_s^d(x) = 0\right\}.$$

The polar "curve" is

$$\mathcal{P} = \left\{(x,s) \in \mathbb{C}^n \times \mathbb{C} \mid \frac{\partial f_s}{\partial x_i}(x) = 0, \forall i \neq k\right\}.$$

In the chart $x_k = 1$ we denote the homogeneization of $\mathcal{P}$ (with s a parameter) by $\bar{\mathcal{C}}$, and the "real" polar curve $\mathcal{C}$ in this chart is the closure of $\bar{\mathcal{C}} \setminus X_\infty$. The part at infinity of $\mathcal{C}$ is $\mathcal{C}_\infty = \mathcal{C} \cap X_\infty$.

Let $B_\infty(s) = \mathrm{pr}_t\{(x:0,t,s) \in \mathcal{C}_\infty\}$. For a generic s', $\mathcal{B}_\infty(s') = B_\infty(s')$. Then the critical parameters for $\mathcal{B}_\infty(s)$ is included in the set of parameters where $\#B_\infty(s)$ fails to be equal to $\#\mathcal{B}_\infty(s')$ (in fact $B_\infty(s)$ may be infinite).
We set $X_* = \{(x:z,c,s) \in X \mid (x:0,c,s) \in \mathcal{C}_\infty\}$, for non-critical parameters it corresponds to union of the irregular fibers at infinity. Now a change of λ corresponds a change in the value of the intersection multiplicity of the polar curve $\mathcal{C}$ with X_*. The critical parameters for λ are given as the projection to the s-coordinate of

$$\overline{(\mathcal{C} \cap X_*) \setminus \mathcal{C}_\infty} \cap (z=0).$$

At last we compute parameters where the cardinal of $\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_\infty(s)$ changes.

5. Examples

5.1. **Briançon polynomial.** The following example shows how to use the program once you have started Singular. We have to load the library `critic.lib`, then we set the ring, with $n+1$ variables, the last variable will able to have the critical values (as the zeroes of a polynomial) in return. The following code gives critical values at infinity of Briançon polynomial.

```
LIB "critic.lib";
ring r = 0, (x,y,t), dp;
poly s = xy+1;
poly p = x*s+1;
poly f = 3*y*p^3+3*p^2*s-5*p*s-s;
crit(f);
```

The result is:

```
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of 3t2+16t
> Milnor number at infinity : 4
> Details of critical values at infinity :
>   t       1
>   3t+16   3
```

This shows, that there is no affine critical value (as the root of the polynomial 1) and that $\mathcal{B}_\infty = \{0, -\frac{16}{3}\}$ (as the root of the polynomial t and $3t+16$) are the critical values at infinity, with Milnor number at infinity respectively equal to 1 and 3.

5.2. **More variables.** Let $f(a,b,c,d) = a + a^4b + b^2c^3 + d^5$ be the example of Choudary-Dimca, [CD] and [ACD]. This polynomial has isolated singularities at infinity. The only singularity is a singularity at infinity for the critical value 0. Let's check it.

```
ring r = 0, (a,b,c,d,t), dp;
poly f = a+a^4*b+b^2*c^3+d^5;
```

```
crit(f);
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
> Milnor number at infinity : 8
```

5.3. **A family.** We give example of deformation, we first need to load the library `defpoly.lib`, then we set a ring in $n+1$ variables, where the last variable is the parameter of the deformation. For instance we consider the deformation $f_s(x,y) = y(1-sx)(y-(s-1)x)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = y*(1-sx)*(y-(s-1)*x);
parCrit(f);
> Critical parameters are included in the roots of s2-s
```

Then the critical parameters are $s=0$ and $s=1$.

5.4. **A trivial family.** Another deformation is $f_s(x,y) = x(x^3y + sx^2 + s^2x + 1)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = x*(x^3*y+s*x^2+s^2*x+1);
parCrit(f);
> Critical parameters are included in the roots of 1
```

Then $\mathfrak{m}(f_s)$ and the degree are constant; by Theorem 2 it implies that for all $s, s' \in \mathbb{C}$, f_s and $f_{s'}$ are topologically equivalent.

5.5. **Combination.** We consider the family $f_s(x,y) = (x - s^2 - 1)(x^2y + 1)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = (x-s^2-1)*(x^2*y+1);
parCrit(f);
> Critical parameters are included in the roots of s2+1
```

For a generic value we have

```
LIB "critic.lib";
ring r = (0,s), (x,y,t), dp;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of t
> Affine Milnor number : 1
> Critical values at infinity are the roots of t+(s2+1)
> Milnor number at infinity : 1
```

And for a critical parameter ($s = i$ or $s = -i$):

```
ring r = (0,s), (x,y,t), dp;
minpoly = s^2+1;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
> Milnor number at infinity : 1
```

References

[ACD] E. Artal-Bartolo, P. Cassou-Noguès and A. Dimca, Sur la topologie des polynômes complexes, Singularities (Oberwolfach, 1996), *Progr. Math.*, 162, Birkhäuser, Basel, 317-343, *1998.*

[BM] G. Bailly-Maître, Monodromies des polynômes de deux variables complexes, *Thèse de l'université de Bordeaux, 2000.*

[Bo] A. Bodin, Invariance of Milnor numbers and topology of complex polynomials, *Comment. Math. Helv.*, 78, 134-152, *2003.*

[BT] A. Bodin and M. Tibăr, Topological equivalence in families of complex polynomials, *preprint.*

[Br] S.A. Broughton, Milnor numbers and the topology of polynomials hypersurfaces, *Inv. Math.*, 92, 217-241, *1988.*

[CD] A. Choudary and A. Dimca, Complex hypersurfaces diffeomorphic to affine spaces, *Kodai Math. J.*, 17, 171-178, *1994.*

[GPS] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 2.0: a computer algebra system for polynomial computations. Centre for computer algebra, university of Kaiserslautern, *2001.* `http://www.singular.uni-kl.de`.

[Ha] Hà H.V., Sur la fibration globale des polynômes de deux variables complexes, *C. R. Acad. Sci. Paris*, 309, 231-234, *1989.*

[HL] Hà H.V. and Lê D.T., Sur la topologie des polynômes complexes, *Acta Mathematica Vietnamica*, 9, 21-32, *1984.*

[LR] Lê D.T. and C.P. Ramanujam, The invariance of Milnor's number implies the invariance of the topological type, *Amer. J. Math.*, 98, 67-78, *1976.*

[Mi] J. Milnor, Singular points of complex hypersurfaces, *Ann. of Math.*, Studies 61, Princeton University Press, *1968.*

[Pa] A. Parusiński, On the bifurcation set of complex polynomial with isolated singularities at infinity, *Compositio Math.*, 97, 369-384, *1995.*

[ST] D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, *Duke Math. J.*, 80, 771-783, *1995.*

[Th] R. Thom, Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.*, 75, 249-312, *1969.*

[Ti] J.G. Timourian, The invariance of Milnor's number implies topological triviality, *Amer. J. Math.*, 99, 437-446, *1977.*

laboratoire AGAT, UFR de Mathématiques, Université Lille I, 59655 Villeneuve d'Ascq

E-mail address: `Arnaud.Bodin@agat.univ-lille1.fr, http://www-gat.univ-lille1.fr/~bodin`